

Flat Connections and Brauer Type Algebras

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Abstract

In this paper, we introduce a Brauer type algebra $B_G(\Upsilon)$ associated with every pseudo reflection group and every Coxeter group G . When G is a Coxeter group of simply-laced type we show $B_G(\Upsilon)$ is isomorphic to the generalized Brauer algebra of simply-laced type introduced by Cohen, Gijsbers and Wales (*J. Algebra*, **280** (2005), 107-153). We also prove $B_G(\Upsilon)$ has a cellular structure and be semisimple for generic parameters when G is a dihedral group or the type H_3 Coxeter group. Moreover, in the process of construction, we introduce a further generalization of Lawrence-Krammer representation to complex braid groups associated with all pseudo reflection groups.

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1 Introduction

Brauer algebras $B_n(\tau)$ introduced by Brauer [Br] are certain algebras connected with representation theory and knot theory. These algebras have natural deformations found by Birman , Wenzl [BW] and by Murakami [Mu], which are called BMW algebras.

Like many objects related to Lie theory, Brauer algebras can be generalized to other general reflection groups (here by a general reflection group we mean a Coxeter group or a pseudo reflection group). In [Ha] Häring-Oldenburg introduced the Cyclotomic Brauer algebras and cyclotomic BMW algebras associated with the $G(m, 1, n)$ type pseudo reflection groups. Slightly later in [CGW1] Cohen, Gijsbers and Wales introduced a Brauer type algebra and a BMW type algebra for each simply laced Coxeter group. It is proved that these new Brauer type algebras share many nice algebraic properties with Brauer algebras like semisimplicity for generic parameters in Cohen-Frenk-Wales [CFW], supporting Cellular structures Cohen-Frenk-Wales op. cit in the sense of Graham-Lehrer [GL]. In [GH] Goodman and Hauschild introduced Affine BMW algebras as a generalization of BMW algebras to affine \hat{A}_n type.

It is asked in Cohen-Gijsbers-Wales [CGW1] whether there exist Brauer type algebras and BMW type algebras for non simply laced Coxeter groups. With the help of KZ connections, we introduce in this paper a Brauer type algebra $B_G(\Upsilon)$ for each general reflection group G . we also justify that the algebras is a suitable candidates for general Brauer type algebra from the following aspects.

- If W_Γ is a simply laced Coxeter group of type Γ , the algebra $B_{W_\Gamma}(\Upsilon)$ coincides with the simply laced Brauer algebra of type Γ introduced in Cohen-Gijsbers-Wales ibid (Definition 8.2, Theorem 8.4). if G is a type $G(m, 1, n)$ pseudo reflection group, the cyclotomic Brauer algebra introduced by Häring-Oldenburg in [Ha] appears as a direct component of our algebra $B_G(\Upsilon)$ (Theorem 8.6).
- When G is a finite pseudo reflection group (including all finite Coxeter groups), $B_G(\Upsilon)$ supports a nicely shaped flat connection on the complementary space of reflection hyperplanes of G . Existence of such a connection is a general phenomenon among finite pseudo reflection groups Broué -Malle-Rouquier [BMR], and simply laced Brauer algebras (Theorem 3.2, Theorem 5.3). These flat connections insure in some sense that $B_G(\Upsilon)$ can be deformed to certain BMW type algebras.
- Every $B_G(\Upsilon)$ induces a generalized Lawrence-Krammer representation of the associated complex braid group A_G (Theorem 5.2).
- When G is finite, $B_G(\Upsilon)$ is a finite dimensional algebra containing $\mathbb{C}G$ (Theorem 5.1). There exists a natural anti-involution in $B_G(\Upsilon)$ (Lemma 5.5) which may be used to construct a cellular structure.
- When G is a dihedral group or a H_3 type Coxeter group, the algebra $B_G(\Upsilon)$ has a cellular structure, and is semisimple for generic Υ (Sections 6-7).

Before giving the definition, we set up some notations which will be used throughout this paper. Let $G \subset U(V)$ be a finite pseudo reflection group. Denote by R the set of pseudo reflections in G , and let $\mathcal{A} = \{H_i\}_{i \in P}$ be the set of reflection hyperplanes. For $s \in R$, define $i(s) \in P$ by requesting $H_{i(s)}$ to be the reflection hyperplane of s . We also denote the reflection hyperplane of s as H_s . Intersection of a subset of \mathcal{A} is called an edge. The action of G on V induces an action of G on \mathcal{A} naturally. For $i, j \in P$, let $R(i, j) = \{s \in R \mid s(H_j) = H_i\}$. For $i \in P$, let G_i be the subgroup of G consisting of elements that fixing H_i pointwise, let $m_i = |G_i|$, and let s_i be the unique element in G_i with exceptional eigenvalue $e^{\frac{2\pi\sqrt{-1}}{m_i}}$.

Set $M_G = V - \cup_{i \in P} H_i$. For $i \in P$, choose a linear function α_i such that $H_i = \ker \alpha_i$ and define $\omega_i = \frac{d\alpha_i}{\alpha_i}$, which are holomorphic closed 1-forms on M_G . We write $s_1 \sim s_2$ for $s_1, s_2 \in R$ if s_1 and s_2 are in the same conjugacy class, and write $i \sim j$ for $i, j \in P$ if $w(H_i) = H_j$ for some $w \in G$. Chose $0 \neq \mu_s \in \mathbb{C}$ for every $s \in R$ and $m_i \in \mathbb{C}$ for every $i \in P$ such that $\mu_{s_1} = \mu_{s_2}$ if $s_1 \sim s_2$, $m_i = m_j$ if $i \sim j$. The data $\{\mu_s, m_i\}_{s \in R, i \in P}$ will be denoted by one symbol Υ . A well-known theorem by Steinberg says that G acts on M_G freely. We denote the group $\pi_1(M_G/G)$, $\pi_1(M_G)$ as A_G, P_G respectively, which are called complex braid groups and complex pure braid groups by many authors.

The original model for above setting came from the symmetric group S_n . First, S_n is realized as a reflection group acting on \mathbb{C}^n by permuting the basis elements, whose reflection hyperplanes are $\{H_{i,j} = \text{Ker}(z_i - z_j)\}_{1 \leq i < j \leq n}$. Then $M_{S_n} = Y_n = \{(z_1, \dots, z_n) \in \mathbb{C}^n \mid z_i \neq z_j \text{ for any } i \neq j\}$, which is the configuration space of n different points on \mathbb{C} . The differential form associated with $H_{i,j}$ is $\frac{dz_i - dz_j}{z_i - z_j}$. The associated group A_G is just the n string braid group B_n .

For $i \neq j$ we denote $H_i \pitchfork H_j$ if $\{k \in P \mid H_i \cap H_j \subset H_k\} = \{i, j\}$. A codimension 2 edge L will be called a crossing edge if there exists $i, j \in P$ such that $H_i \pitchfork H_j$ and $L = H_i \cap H_j$, otherwise L will be called a noncrossing edge.

Definition 1.1. *The algebra $B_G(\Upsilon)$ associated with pseudo reflection group (V, G) is generated by the set $\{T_w\}_{w \in G} \cup \{e_i\}_{i \in P}$ which satisfies the following relations.*

- (0) $T_{w_1} T_{w_2} = T_{w_3}$ if $w_1 w_2 = w_3$.
- (1) $T_{s_i} e_i = e_i T_{s_i} = e_i$, for $i \in P$.
- (1)' $T_w e_i = e_i T_w = e_i$, for $w \in G$ such that $w(H_i) = H_i$, and $H_i \cap V_w$ is a noncrossing edge. Where $V_w = \{v \in V \mid w(v) = v\}$.
- (2) $e_i^2 = m_i e_i$.
- (3) $T_w e_j = e_i T_w$, if $w \in G$ satisfies $w(H_j) = H_i$.
- (4) $e_i e_j = e_j e_i$, if $H_i \pitchfork H_j$.
- (5) $e_i e_j = (\sum_{s \in R(i,j)} \mu_s T_s) e_j = e_i (\sum_{s \in R(i,j)} \mu_s T_s)$, if $H_i \cap H_j$ is a noncrossing edge, and $R(i, j) \neq \emptyset$.

(6) $e_i e_j = 0$, if $H_i \cap H_j$ is a noncrossing edge, and $R(i, j) = \emptyset$.

Each one of these relations can be thought of as generalization of certain relation in the Brauer algebra $B_n(\tau)$. Relation (4) is the generalization of $e_{i,j} e_{k,l} = e_{k,l} e_{i,j}$ for different i, j, k, l , where $e_{i,j}$ is explained in the following Figure 1. Relation (6) can be seen as a special case of relation (5), they are generalizations of the relation $e_{i,j} e_{j,k} = s_{i,k} e_{j,k} = e_{i,j} s_{i,k}$ in $B_n(\tau)$. Where $s_{i,j}$ is the (i, j) permutation. Relation (1)' resembles relation (o) in Definition 2.1 of cyclotomic Brauer algebras. Motivation of introducing these algebras is as follows.

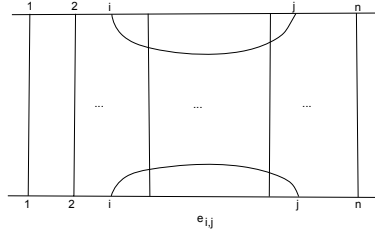


Figure 1: The element $e_{i,j}$

It is well-known that the group algebras of Coxeter groups and pseudo reflection groups have deformations called Hecke algebras. It is possibly less well-known that for any finite Coxeter group or any pseudo reflection group, the infinitesimal deformation to the corresponding Hecke algebra can be described by a KZ connection with nice shape Cherednik [Che] Broué -Malle-Rouquier [BMR].

When G is a finite Coxeter group, the KZ connection Ω_G (Ω'_G) describing deformation of G to the G -type Hecke algebra with equal parameter $H_G(q)$ (G -type Hecke algebra with unequal parameters $H_G(\bar{q})$) is:

$$\Omega_G = \kappa \sum_{i \in P} s_i \omega_i, \quad \Omega'_G = \sum_{i \in P} \kappa_i s_i \omega_i, \quad (1)$$

where κ, κ_i 's are constants such that $\kappa_i = \kappa_j$ if $i \sim j$. Ω_G is a formal connection on M_G . Suppose (U, ρ) is any representation of G . Then Ω_G gives a G -invariant, flat connection $\rho(\Omega_G) = \kappa \sum_{i \in P} \rho(s_i) \omega_i$ on the bundle $M_G \times U$, which further induces a flat connection on the quotient bundle $M_G \times_G U$ whose monodromy representation factors through $H_G(q)$ for suitable q . As a special case, the KZ connection for symmetric group S_n is

$$\Omega_n = \kappa \sum_{1 \leq i < j \leq n} s_{i,j} \frac{dz_i - dz_j}{z_i - z_j}. \quad (2)$$

The KZ connection for a pseudo reflection group G has the form:

$$\Omega_G = \sum_{i \in P} \left(\sum_{s: i(s)=i} \mu_s s \right) \omega_i, \quad (3)$$

where μ_s are constants such that $\mu_s = \mu_{s'}$ if $s \sim s'$. It plays an important role in Broué - Malle-Rouquier [BMR] in construction of the generalized Hecke algebras for pseudo reflection

groups (see also Ariki-Koike [AK]). Note that in above connections the operator terms come from pseudo reflection group, the flatness and G-equivariance come from relations in G.

Now the Brauer algebra $B_n(\tau)$ has also a natural deformation $B_n(\tau, l)$, the BMW algebra. In [Ma1], the following flat formal connection supported by $B_n(\tau)$ was implied :

$$\bar{\Omega}_n = \kappa \sum_{1 \leq i < j \leq n} (s_{i,j} - e_{i,j}) \omega_{i,j}.$$

Where $s_{i,j}$ is the (i, j) permutation, $e_{i,j}$ is the element described by above Figure 1. This connection $\bar{\Omega}_n$ is flat and S_n -equivariant (Proposition 4 of [Ma1], see also Proposition 3.1). Marin also proved if (U, ρ) is any representation of $B_n(\tau)$, then the S_n -invariant, flat connection $\rho(\bar{\Omega}_n)$ on the bundle $M_{S_n} \times U$ induces a flat connection on the quotient bundle $M_{S_n} \times_{S_n} U$, whose monodromy representation factor through the BMW algebra $B_n(\tau, l)$ for suitable τ, l (Proposition 4 of [Ma1], see also Theorem 3.2). So the connection $\bar{\Omega}_n$ can be seen as the KZ connection for $B_n(\tau)$.

Later we will show (Theorem 5.3) the deformation of a simply laced Brauer algebra $B_\Gamma(m)$ to the simply laced BMW algebra can be described by a flat connection

$$\bar{\Omega}_\Gamma = \kappa \sum_{i \in P} (s_i - e_i) \omega_i, \quad (4)$$

where $\{e_i\}_{i \in P} \subset B_\Gamma(m)$ is a set of semi-idempotents (by a semi-idempotent we mean elements x satisfying $x^2 = \lambda x$) in one-to-one correspondence with P , the set of reflection hyperplanes.

Suppose Γ is any finite type Dynkin diagram, Denote by W_Γ the Coxeter group of type Γ . Now we present a bold but reasonable hypothesis about the general Brauer type algebra $B_{W_\Gamma}(\Upsilon)$: $B_{W_\Gamma}(\Upsilon)$ can be deformed to certain BMW type algebra and the deformation can be described by a nicely shaped KZ connection $\bar{\Omega}_\Gamma$ on M_{W_Γ} .

The most natural form of the connection $\bar{\Omega}_\Gamma$ is as equation (4). When the set of reflections in W_Γ contain more than one conjugacy class, in view of the connection Ω'_Γ in Equation (1), the KZ connection associated with $B_{W_\Gamma}(\Upsilon)$ should have a more general form containing more parameters. Thus we make the following hypothesis about $B_{W_\Gamma}(\Upsilon)$.

Hypothesis 1. $B_{W_\Gamma}(\Upsilon)$ contains the group algebra $\mathbb{C}W_\Gamma$ and a set of semi-idempotents $\{e_i\}_{i \in P}$, such that the formal connection $\bar{\Omega}_\Gamma = \sum_{i \in P} \kappa_i (s_i - e_i) \omega_i$ is flat and W_Γ -invariant, where $\{\kappa_i\}_{i \in P}$ is a set of constant numbers such that $\kappa_i = \kappa_j$ if $i \sim j$. More over, $B_{W_\Gamma}(\Upsilon)$ is generated by $W_\Gamma \cup \{e_i\}_{i \in P}$.

More generally for a pseudo reflection group G , we assume the algebra $B_G(\Upsilon)$ should also contain $\mathbb{C}G$ and a set of special elements $\{e_i\}_{i \in P}$, such that the formal connection $\bar{\Omega}_G = \sum_{i \in P} (\sum_{s: i(s)=i} \mu_s s - e_i) \omega_i$ is flat and G -invariant. The shape of $\bar{\Omega}_G$ is also inspired by the generalized Lawrence-Krammer representation of A_G defined later.

By Theorem 3.1 in Kohno [Ko1], we can derive some algebraic relations between $R \cup \{e_i\}_{i \in P}$ from flatness and G -invariance of $\bar{\Omega}_G$. But these relations are not enough to produce the Brauer type algebra we want.

Another common feature of simply laced BMW algebras as proved in Cohen-Gijsbers-Wales [CGW1] is that they all contain the generalized Lawrence-Krammer representations of simply laced Artin groups introduced by Cohen-Wales [CW] and Digne [Di]. Recently Marin [Ma2] introduced a generalized Lawrence-Krammer representations of A_G for any complex reflection group G (in this paper we use the phrase 'complex reflection group' to denote those pseudo reflection groups all of whose pseudo reflections have degree two). In section 4 we introduce a slightly further generalization of the Lawrence-Krammer representation to A_G for each pseudo reflection group G . The idea is as follows. Let $V_G = \mathbb{C}\{v_i\}_{i \in P}$ be a vector space with a basis in one-to-one correspondence with $\{H_i\}_{i \in P}$. Action of G on $\{H_i\}_{i \in P}$ induces a natural representation $\iota : G \rightarrow \text{End}(V_G)$.

From Marin's work [Ma3] we see that the simply laced type Lawrence-Krammer representations can be described by certain flat, G -invariant connection

$$\Omega_{LK} = \kappa \sum_{i \in P} (\iota(s_i) - p_i) \omega_i \quad (5)$$

on the bundle $M_G \times V_G$. Where $s_i \in R$ is the unique pseudo reflection having H_i as its reflection hyperplane. We observe that for any i the map $p_i \in \text{End}(V_G)$ is a projector to the line $\mathbb{C}v_i \subset V_G$. It inspires us to consider a special kind of connection on the bundle $M_G \times V_G$ for any pseudo reflection group G :

$$\Omega_{LK} = \sum_{i \in P} \left(\sum_{s: \iota(s)=i} \mu_s \iota(s) - p_i \right) \omega_i$$

where μ_s 's are constants such that $\mu_s = \mu_{s'}$ if $s \sim s'$. And $p_i \in \text{End}(V_G)$ is a projector to the line $\mathbb{C}v_i \subset V_G$ for any i . Explicitly suppose $p_i(v_j) = \alpha_{i,j}v_i$ ($j \neq i$) and $p_i(v_i) = m_i v_i$. Then we have

Theorem 4.2 *The connection Ω_{LK} is flat and G -equivariant if and only if the following two conditions hold: a) $\alpha_{i,j} = \sum_{s: \iota(s)=v_j} \mu_s$; b) $m_i = m_j$ if $i \sim j$.*

When Ω_{LK} satisfy the conditions in Theorem 4.2, it induces a flat connection $\bar{\Omega}_{LK}$ on the quotient bundle $M_G \times_G V_G$. We define the generalized Lawrence-Krammer representation of A_G as the monodromy representation of $\bar{\Omega}_{LK}$. When G is a complex reflection group, and $\mu_s = 1$ for all s , the connection Ω_{LK} becomes the flat connection of Marin [Ma2].

Now suppose $\{p_i\}_{i \in P} \subset \text{End}(V_G)$ satisfy conditions in Theorem 4.2. It is proved in Cohen-Gijsbers-Wales [CGW1] that every simply laced BMW algebra contain a generalized Lawrence-Krammer representation, just as the case of braid groups in Zinno [Z]. This fact can be explained in infinitesimal level in the following sense. Defining a map $\phi : W_\Gamma \cup \{e_i\}_{i \in P} \rightarrow \text{End}(V_G)$ by $\phi(w) = \iota(w)$ for $w \in W_\Gamma$; $\phi(e_i) = p_i$ for $i \in P$, then ϕ can be extended to a representation $B_\Gamma(\tau) \rightarrow \text{End}(V_G)$.

Regarding to these facts we make another hypothesis about $B_G(\Upsilon)$:

Hypothesis 2. *For any pseudo reflection group G , the map $w \mapsto \iota(w)$ for $w \in G$; $e_i \mapsto p_i$ for $i \in P$ can be extended to a representation $B_G(\Upsilon)$. Where we suppose P_i satisfy conditions in above Theorem 4.2.*

Now we search if there exist suitable relations between G and $\{e_i\}_{i \in P}$ such that the resulted algebra $B_G(\Upsilon)$ satisfy Hypotheses 1 and 2. As a result we find there do exists one, the algebra defined in Definition 1.1 satisfy these two conditions quite nicely.

Proposition 5.1 *The connection $\bar{\Omega}_G = \sum_{i \in P} (\sum_{s: i(s)=i} \mu_s T_s - e_i) \omega_i$ are flat and G -invariant. Where T_s, e_i are as in Definition 1.1.*

Theorem 5.2 *Using notations in section 4. The map $w \mapsto \iota(w)$, $e_i \mapsto p_i$ extends a representation $B_G(\Upsilon) \rightarrow \text{End}(V_G)$.*

In fact we believe it is the best choice. There are two other slightly different choices: take off the relation (1)' or weaken relation (6). In the last section we explain some reason of choosing Definition 5.1.

In section 8 we show that there exist canonical presentations for $B_G(\Upsilon)$ when G is a finite Coxeter group or a type $G(m, 1, n)$ pseudo reflection group. (Definition 8.1, Definition 8.2, Theorem 8.4). They can be seen a generalization of the presentation for simply laced Brauer algebras in [CFW]. Definition 8.2 can be naturally generalized to the cases when G is an infinite type Coxeter group (Remark 8.1). In a canonical presentation each node i of the Dynkin diagram corresponds to a pair of generators $\{s_i, e_i\}$. Thus in cases of $G(m, 1, n)$ type we have one more generator e_0 than in the canonical presentation of cyclotomic Brauer algebras $\mathcal{B}_{m,n}(\sigma)$. It is this new generator e_0 that making $B_G(\Upsilon)$ slightly larger than $\mathcal{B}_{m,n}(\sigma)$ (Theorem 9.1). Through canonical presentations we see immediately that $B_G(\Upsilon)$ coincides with the simply laced Brauer algebra of [CGW1] if G is a simply laced type Coxeter group. These canonical presentations may be helpful to define new BMW type algebras, which will be discussed in a future paper.

Section 6 and 7 are devoted to show these new algebras for non simply laced Dynkin diagrams are indeed interesting objects by finding some nice algebraic properties of them. Concretely they are: **(SEM)** semisimple for generic parameters; **(CEL)** having cellular structures; **(DEF)** deformability and **(STA)** dimension stability (having the same dimension for any parameters Υ). Because of limitations of spaces in this paper we only study in detail the cases when G is a dihedral group or the H_3 type Coxeter group.

When G is one of above mentioned cases, we prove that $B_G(\Upsilon)$ satisfies **(SEM)**, **(CEL)** and **(STA)** and write down the condition for $B_G(\Upsilon)$ to be semisimple . Through the study of the H_3 case we find for the H_3 type Artin group three new 15 dimensional irreducible representations except for the generalized Lawrence-Krammer representation, and one new 5 dimensional irreducible representation. All of these representations have clear combinatorial meaning, they are related to two kinds of natural actions of W_{H_3} on certain sets. We believe that the existence of the KZ connections supports the property **(DEF)** for every $B_G(\Upsilon)$.

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2 Preliminaries

2.1 Brauer type algebras and BMW type algebras

Brauer algebra $B_n(\tau)$ is a graphic algebra in the sense that it has a basis consisting of elements presented by graphs, and the relations between them can be described through graphs. $B_n(\tau)$ has a canonical presentation with generators s_1, \dots, s_{n-1} , e_1, \dots, e_{n-1} and relations listed in table 1. $B_n(\tau)$ has a natural deformation discovered by Birman, Murakami, Wenzl which are now called BMW algebras [BW] [Mu]. These algebras support a Markov trace which gives the Kauffman polynomial invariants of Links. We denote these BMW algebras as $B_n(\tau, l)$. Where l is a parameter of deformation. There is $B_n(\tau) \cong B_n(\tau, 1)$. We list generators and relations of $B_n(\tau)$ and $B_n(\tau, l)$ in the following table according to [CGW1]. Where $m = \frac{l-1}{1-\tau}$.

The structure of Brauer algebras and BMW algebras are studied extensively in last 20 years. See for example [W] [RH]. They have the following basic properties.

Theorem (Wenzl) Let the ground ring be a field of character 0, then $B_n(\tau)$ is semisimple if and only if $\tau \notin \mathbb{Z}$ or $\tau \in \mathbb{Z}$ and $\tau > n$.

TABLE 1. Presentation for $B_n(\tau)$.

	$B_n(\tau)$	$B_n(\tau, l)$
Generators	$s_1, \dots, s_{n-1}; e_1, \dots, e_{n-1}$	$X_1, \dots, X_{n-1}; E_1, \dots, E_{n-1}$
Relations	$s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1}$ for $1 \leq i \leq n-2$; $s_i s_{i-1} e_i = e_{i-1} s_i s_{i-1}$ for $2 \leq i \leq n-1$; $s_i s_{i+1} e_i = e_{i+1} s_i s_{i+1}$ for $1 \leq i \leq n-2$; $s_i s_j = s_j s_i, i-j \geq 2$; $s_i^2 = 1$, for all i ; $s_i e_i = e_i$, for all i ; $e_i s_{i+1} e_i = e_i, 1 \leq i \leq n-2$; $e_i s_{i-1} e_i = e_i, 2 \leq i \leq n-1$; $s_i e_j = e_j s_i, i-j > 1$; $e_i^2 = \tau e_i$, for all i .	$X_i X_{i+1} X_i = X_{i+1} X_i X_{i+1}$ for $1 \leq i \leq n-2$; $X_i X_{i-1} E_i = E_{i-1} X_i X_{i-1}$ for $2 \leq i \leq n-1$; $X_i X_{i+1} E_i = E_{i+1} X_i X_{i+1}$ for $1 \leq i \leq n-2$; $X_i X_j = X_j X_i, i-j \geq 2$; $l(X_i^2 + mX_i - 1) = mE_i$, for all i ; $X_i E_i = l^{-1} E_i$, for all i ; $E_i X_{i+1} E_i = l E_i, 1 \leq i \leq n-2$; $E_i X_{i-1} E_i = l E_i, 2 \leq i \leq n-1$; $X_i E_j = E_j X_i, i-j > 1$; $E_i^2 = \tau E_i$.

Semisimplicity condition for any groundrings is obtained by Rui [RH]. Many algebras related to Lie theory have cellular structures in the sense of Graham and Lehrer [GL]. We

recall the definition of a cellular structure (cellular algebra). In the same paper Graham and Lehrer proved Brauer algebras support cellular structures. Similar result for BMW algebras are proved by Xi [Xi2].

Definition (Graham, Lehrer)[15] A cellular algebra over R is an associative algebra A , together with cell datum $(\Lambda, M, C, *)$ where

- (C1) Λ is a partially ordered set and for each $\lambda \in \Lambda$, $M(\lambda)$ is a finite set such that $C : \prod_{\lambda \in \Lambda} M(\lambda) \times M(\lambda) \rightarrow A$ is an injective map with image an R -basis of A .
- (C2) If $\lambda \in \Lambda$ and $S, T \in M(\lambda)$, write $C(S, T) = C_{S,T}^\lambda \in A$. Then $*$ is an R -linear anti-involution of A such that $*(C_{S,T}^\lambda) = C_{T,S}^\lambda$.
- (C3) If $\lambda \in \Lambda$ and $S, T \in M(\lambda)$ then for any element $a \in A$ we have

$$a C_{S,T}^\lambda \equiv \sum_{S' \in M(\lambda)} r_a(S', S) C_{S',T}^\lambda \pmod{A(< \lambda)}$$
Where $r_a(S', S) \in R$ is independent of T and where $A(< \lambda)$ is the R -submodule of A generated by $\{C_{S'',T''}^\mu | \mu < \lambda; S'', T'' \in M(\mu)\}$.

The cyclotomic Brauer algebras $\mathcal{B}_{m,n}(\delta)$ of Häring-Oldenburg has the following presentation. (borrowed from [RX])

Definition 2.1. *The algebra $\mathcal{B}_{m,n}(\delta)$ is generated by a set $\{s_i, e_i\}_{1 \leq i \leq n} \cup \{t_j\}_{1 \leq j \leq n}$ with the following relations.*

- | | |
|---|--|
| a) $s_i^2 = 1$, for $1 \leq i \leq n$. | k) $e_i s_i = e_i = s_i e_i$, for $1 \leq i \leq n-1$. |
| b) $s_i s_j = s_j s_i$, if $ i-j > 1$. | l) $s_i e_{i+1} s_i = s_{i+1} e_i$, for $1 \leq i \leq n-2$. |
| c) $s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1}$, for $1 \leq i < n-1$. | m) $e_{i+1} e_i s_{i+1} = e_{i+1} s_i$,
for $1 \leq i \leq n-2$. |
| d) $s_i t_j = t_j s_i$, if $j \neq i, i+1$. | n) $e_i e_j e_i = e_i$, if $ i-j = 1$. |
| e) $e_i^2 = \delta_0 e_i$, for $1 \leq i < n$. | o) $e_i t_i t_{i+1} = e_i = t_i t_{i+1} e_i$,
for $1 \leq i < n$. |
| f) $s_i e_j = e_j s_i$, if $ i-j > 1$. | p) $e_i t_i^a e_i = \delta_a e_i$, for $1 \leq a \leq m-1$
$1 \leq i \leq n-1$. |
| g) $e_i e_j = e_j e_i$, if $ i-j > 1$. | q) $t_i^m = 1$, for $1 \leq i \leq n$. |
| h) $e_i t_j = t_j e_i$, if $j \neq i, i+1$. | |
| i) $t_i t_j = t_j t_i$, for $1 \leq i, j \leq n$. | |
| j) $s_i t_i = t_{i+1} s_i$, for $1 \leq i < n$. | |

The subset of generators $\{s_i\}_{1 \leq i \leq n} \cup \{t_j\}_{1 \leq j \leq n}$ together with relations (a),(b),(c),(d),(i),(j),(q) generate the cyclotomic reflection group of type $G(m, 1, n)$ whose group algebra is imbedded in $\mathcal{B}_{m,n}(\delta)$. The original paper [Ha] define more complicated cyclotomic BMW algebra, where the generators and relations can be represented by graphs also. These algebras have many properties parallel with Brauer algebras. In [RX], the authors proved they are semisimple and classified their irreducible representations under certain generic conditions. By Goodman in [Go] and by Yu in [Yu] independently, $\mathcal{B}_{m,n}(\delta)$ are shown to have cellular structures.

Finite type simply-laced Dynkin diagram consists of ADE type Dynkin diagrams. For every such Dynkin diagram Γ , the following table are presentation for algebra $B_\Gamma(\tau)$ and algebra $B_\Gamma(\tau, l)$ defined in [CGW1]. When Γ is A_{n-1} , it is straightforward to see they coincide with $B_n(\tau)$ and $B_n(\tau, l)$ respectively. Let I be the set of nodes of Γ . When $i, j \in I$ are connected by an edge we write $i \sim j$. Otherwise we write $i \approx j$. Set $m = \frac{l-l^{-1}}{1-\tau}$.

The simply laced Brauer algebras have no graph to representing their elements any more, but they have almost all important algebraic properties of Brauer algebras. In [CFW] the authors proved when Γ is finite ADE type, these algebras $B_\Gamma(\tau)$ are free module over $\mathbb{Z}[\tau^{\pm 1}]$, and be semisimple after tensored with $\mathbb{Q}(\tau)$.

TABLE 2. Presentation for $B_\Gamma(\tau)$.

	$B_\Gamma(\tau)$	$B_\Gamma(\tau, l)$
Generators	$s_i (i \in I); e_i (i \in I)$	$X_i (i \in I); E_i (i \in I)$
Relations	$s_i s_j s_i = s_j s_i s_j$, if $i \sim j$; $s_i s_j e_i = e_j s_i s_j$ if $i \sim j$; $s_i s_j = s_j s_i$, if $i \approx j$; $s_i^2 = 1$, for all i ; $s_i e_i = e_i$, for all i ; $e_i s_j e_i = e_i$, if $i \sim j$; $s_i e_j = e_j s_i$, if $i \approx j$; $e_i^2 = \tau e_i$, for all i .	$X_i X_j X_i = X_j X_i X_j$, if $i \sim j$; $X_i X_j E_i = E_j X_i X_j$ if $i \sim j$; $X_i X_j = X_j X_i$, if $i \approx j$; $l(X_i^2 + mX_i - 1) = mE_i$, for all i ; $X_i E_i = l^{-1}E_i$, for all i ; $E_i X_j E_i = lE_i$, if $i \sim j$; $X_i E_j = E_j X_i$, if $i \approx j$; $E_i^2 = \tau E_i$.

2.2 Pseudo reflection groups, Complex braid groups and Hecke algebras

Let V be a complex linear space. An element s in $GL(V)$ is called a pseudo reflection if it can be presented as $\text{diag}(\xi, 1, \dots, 1)$ under some basis of V , where ξ is a root of unit. We call ξ as the exceptional eigenvalue of s . If ξ is -1 then s is simply called a reflection. A finite group $G \subset GL(V)$ is called a pseudo reflection group if it is generated by pseudo reflections. If G is generated by reflections then we call it a complex reflection group. When V is an irreducible representation of G , (V, G) is called an irreducible pseudo reflection group. Every pseudo reflection group is isomorphic to direct product of some irreducible factors. Isomorphism class of irreducible pseudo reflection groups are classified by Shephard-Todd [ST]. They consists of an infinite family $\{ G(m, p, n) \}$ ($n \leq 1, m \leq 2, p|m$) and 34 exceptional ones.

For a pseudo reflection group (V, G) we assume notations in section 1. Denote $\pi_1(M_G)$ as P_G , then there is an exact sequence: $1 \rightarrow P_G \rightarrow A_G \rightarrow G \rightarrow 1$.

By Ariki, Koike in [AK] and by Broue-Malle-Rouquier in [BMR], there exists a Hecke algebra $H_G(\bar{\lambda})$ associated with any pseudo reflection group G , where $\bar{\lambda}$ is a set of parameters. The Hecke algebra $H_G(\bar{\lambda})$ is a quotient algebra of the group algebra $\mathbb{C}A_G$. For most G 's, we have $\dim H_G(\bar{\lambda}) = |G|$, and this relation is a conjecture for other cases. For some G 's and for generic $\bar{\lambda}$, $H_G(\bar{\lambda})$ is a semisimple algebra whose irreducible representations are in one to

one correspondence with those of G in a natural way. This correspondence can be described by the following KZ connection.

Suppose $\{\mu_s\}_{s \in R}$ is a set of constants satisfying the condition: $\mu_{s_1} = \mu_{s_2}$ if s_1 is conjugate to s_2 . Here for simplicity we choose a connection with slightly different appearance from [BMR].

Proposition 2.1 (Broué -Malle-Rouquier [BMR]). *The formal connection*

$$\Omega_G = \sum_{i \in P} \left(\sum_{s \in R, i(s)=i} \mu_s s \right) \omega_i$$

on $M_G \times \mathbb{C}G$ is flat and G -invariant.

Now suppose $\rho : G \rightarrow GL(U)$ is a representation of G on a complex linear space U . The group G acts on the bundle $M_G \times U$ as: $g(p, u) = (g \cdot p, \rho(g)(u))$ for $g \in G$, $p \in M_G$ and $u \in U$. The quotient space $M_G \times U/G$ become a linear bundle over M_G/G naturally, and it will be denoted as $M_G \times_G U$. Now suppose $\Omega = \sum_{i \in P} X_i \omega_i$ is a connection on $M_G \times U$, where $X_i \in \text{End}(U)$ for any i . Here is the condition for Ω to induce a connection on $M_G \times_G U$. See section 4 of [BMR] for some backgrounds about connections.

Proposition 2.2. *The connection Ω induce a connection on $M_G \times_G U$ if :*

$$\rho(w)X_i\rho(w)^{-1} = X_{w(i)} \text{ for any } w \in G \text{ and } i \in P.$$

When the condition in above proposition is satisfied, we call Ω as a G -invariant connection. Suppose (E, ρ) is a linear representation of G . By above proposition

$$\rho(\Omega_G) = \kappa \sum_{i \in P} \left(\sum_{s \in R, i(s)=i} \mu_s \rho(s) \right) \omega_i$$

defines a flat connection on the bundle $M_G \times E$. It induces a flat connection $\bar{\Omega}_\rho$ on the quotient bundle $M_G \times_G E$ because of G -invariance of $\rho(\Omega_G)$. By taking monodromy a family of representations of A_G parameterized by (κ, μ_s) are obtained. It is proved in [BMR] that for generic κ these monodromy representations factor through $H_G(\bar{\lambda})$ for suitable $\bar{\lambda}$.

The following theorem from Marin [Ma2](Theorem 2.9) will be used in the following sections. Let $G \subset GL(V)$ be a pseudo reflection group. Let \mathcal{A} , P , ω_i , M_G , P_G , A_G be defined as in Section 1. Suppose $I \subset V$ be a complex line. The maximal parabolic subgroup G_0 of G associated with I is the subgroup of G formed by elements which stabilize I pointwise. By Steinberg's theorem, G_0 is generated by reflections R_0 of G whose reflecting hyperplane contains I . We set $\mathcal{A}_0 = \{H_i \in \mathcal{A} | I \subset H_i\}$, and $P_0 = \{i \in P | I \subset H_i\}$. Let $M_0 = V - \cup_{i \in P_0} H_i$. Since G_0 is a pseudo reflection group, it has associated braid group A_{G_0} and pure braid group P_{G_0} . It is clear we have identifications: $P_{G_0} \cong \pi_1(M_0)$, $A_{G_0} \cong \pi_1(M_0/G_0)$. Following [BMR], P_{G_0} and A_{G_0} can be imbedded into P_G and A_G in the following way, whose image are called maximal parabolic subgroup of P_G , respectively A_G .

We endow V with a G -invariant unitary form and denote the associated norm as $\| \cdot \|$. Let $x_1 \in I$ such that $x_1 \notin H$ for any $H \in \mathcal{A} \setminus \mathcal{A}_0$. There exists $\epsilon > 0$ such that, for all $x \in V$

with $\|x - x_1\| \leq \epsilon$ we have $x \notin H$ for all $H \in \mathcal{A} \setminus \mathcal{A}_0$. Let $D = \{x \in V \mid \|x - x_1\| \leq \epsilon\}$. It is easy to see the natural morphism $\pi_1(M_G \cap D) \rightarrow \pi_1(M_0)$ is an isomorphism, hence the natural inclusion $\pi_1(M_G \cap D) \rightarrow \pi_1(M_G)$ defines an embedding $P_{G_0} \rightarrow P_G$. Since D is setwise stabilized by G_0 , this embedding extends to an embedding $A_{G_0} \rightarrow A_G$. It is proved in [BMR] that such embeddings are well-defined up to P_G -conjugation.

Now suppose on a bundle $M_G \times E$ there is a flat connection $\Omega = \kappa \sum_{i \in P} X_i \omega_i$. Denote the monodromy representation of P_G resulted from Ω as ρ . If Ω is G -invariant, denote the monodromy representation of A_G resulted from Ω as $\tilde{\rho}$. Looking P_{G_0} , A_{G_0} as parabolic subgroups of P_G , A_G , we obtain representations of P_{G_0} and A_{G_0} by restriction of ρ and $\tilde{\rho}$ respectively. On the other hand, we define a connection on M_0 : $\Omega_0 = \kappa \sum_{i: I \subset H_i} X_i \omega_i$. A simple discussion by using Theorem 3.1 of Section 3 shows Ω_0 is also flat. We denote the monodromy representation of P_{G_0} resulted from Ω_0 as ρ_0 . When Ω is G -invariant, then Ω_0 is G_0 -invariant. In these cases we denote the monodromy representation of A_{G_0} resulted from Ω_0 as $\tilde{\rho}_0$. The following theorem is proved in Marin [Ma2] (Theorem 2.9).

Theorem 2.1. *For generic κ , the P_{G_0} representation ρ_0 is isomorphic to the restriction of ρ . When Ω is G -invariant, the A_{G_0} representation $\tilde{\rho}_0$ is isomorphic to the restriction of $\tilde{\rho}$.*

3 Flat connections for BMW algebras

We begin with some knowledge for hyperplane arrangements. Let E be a complex linear space. An hyperplane arrangement (or arrangement simply) in E means a finite set of hyperplanes contained in E . Let $\mathcal{A} = \{H_i\}_{i \in I}$ be an arrangement in E , we denote the complementary space $E - \cup_{i \in I} H_i$ as $M_{\mathcal{A}}$. Intersection of any subset of \mathcal{A} is called an edge. If L is an edge of \mathcal{A} , define $\mathcal{A}_L = \{H_i \in \mathcal{A} \mid L \subset H_i\} = \{H_i\}$ and $I_L = \{i \in I \mid L \subset H_i\}$.

For every $i \in I$, chose a linear form f_i with kernel H_i . Set $\omega_i = d \log f_i$, which is a holomorphic closed 1-form on $M_{\mathcal{A}}$. Consider the formal connection $\Omega = \kappa \sum_{i \in I} X_i \omega_i$. Here X_i are linear operators to be determined. When we take X_i 's as endomorphisms of some linear space E , then Ω is realized as a connection on the bundle $M_{\mathcal{A}} \times E$. We have the following theorem of Kohno.

Theorem 3.1 (Kohno [Ko1]). *The formal connection Ω is flat if and only if:*

$[X_i, \sum_{j \in I_L} X_j] = 0$ for any codimension 2 edge L of \mathcal{A} , and for any $i \in I_L$. Where $[A, B]$ means $AB - BA$.

The following lemma can be proved directly by using graphs.

Lemma 3.1. *In the Brauer algebra $B_n(\tau)$, let $s_{i,j} \in S_n \subset B_n(\tau)$ be (i, j) permutation., let $e_{i,j}$ be as in introduction. we have*

- (1) $e_{i,j} s_{k,l} = s_{k,l} e_{i,j}$ if $\{i, j\} \cap \{k, l\} = \emptyset$;
- (2) $e_{i,j} e_{k,l} = e_{k,l} e_{i,j}$ if $\{i, j\} \cap \{k, l\} = \emptyset$;

$$(3) \ e_{i,j} = e_{j,i};$$

$$(4) \ e_{i,j}e_{i,k} = s_{j,k}e_{i,k} = e_{i,j}s_{j,k}, \text{ for any different } i, j, k;$$

$$(5) \ e_{i,j}^2 = \tau e_{i,j}, \text{ for any } i \neq j;$$

$$(6) \ s_{i,j}e_{j,k} = e_{i,k}s_{i,j}.$$

For $1 \leq i < j \leq n-1$, define $\omega_{i,j} = d(z_i - z_j)/(z_i - z_j)$. Consider the formal connection $\bar{\Omega}_n = \kappa \sum_{i < j} (s_{i,j} - e_{i,j})\omega_{i,j}$. We have

Proposition 3.1 (Marin[Ma1]). *The formal connection $\bar{\Omega}_n$ is flat and S_n invariant.*

Proof. We certify $\bar{\Omega}_n$ satisfies conditions of theorem 3.1. For the arrangement \mathcal{A}_n , there are then following two type of codimension 2 edges

Case 1. $L = H_{i,j} \cap H_{k,l}$, $\{i, j\} \cap \{k, l\} = \emptyset$.

Whence $\mathcal{A}_L = \{H_{i,j}, H_{k,l}\}$. Now we have $s_{a,b}s_{c,d} = s_{c,d}s_{a,b}$ and $e_{a,b}e_{c,d} = e_{c,d}e_{a,b}$ if $\{a, b\} \cap \{c, d\} = \emptyset$. They are most easily seen by using graphs. so $[s_{i,j} - e_{i,j}, s_{k,l} - e_{k,l}] = 0$. Which gives $[s_{i,j} - e_{i,j}, s_{i,j} - e_{i,j} + s_{k,l} - e_{k,l}] = 0 = [s_{k,l} - e_{k,l}, s_{i,j} - e_{i,j} + s_{k,l} - e_{k,l}]$.

Case 2. $L = H_{i,j} \cap H_{j,k}$, where i, j, k are different. In this case $\mathcal{A}_L = \{H_{i,j}, H_{j,k}, H_{i,k}\}$,

$$\begin{aligned} & [s_{i,j} - e_{i,j}, s_{i,k} - e_{i,k} + s_{j,k} - e_{j,k}] \\ &= [s_{i,j}, s_{i,k} + s_{j,k}] + (-e_{i,j}s_{i,k} + e_{i,j}e_{j,k}) + (s_{i,k}e_{i,j} - e_{j,k}e_{i,j}) \\ &+ (-e_{i,j}s_{j,k} + e_{i,j}e_{i,k}) + (s_{j,k}e_{i,j} - e_{i,k}e_{i,j}) + [s_{i,j}, -e_{i,k} - e_{j,k}] \\ &= (-e_{i,j}s_{i,k} + e_{i,j}e_{j,k}) + (s_{i,k}e_{i,j} - e_{j,k}e_{i,j}) \\ &+ (-e_{i,j}s_{j,k} + e_{i,j}e_{i,k}) + (s_{j,k}e_{i,j} - e_{i,k}e_{i,j}) + [s_{i,j}, -e_{i,k} - e_{j,k}] \\ &= (-e_{i,j}s_{i,k} + e_{i,j}e_{j,k}) + (s_{i,k}e_{i,j} - e_{j,k}e_{i,j}) + (-e_{i,j}s_{j,k} + e_{i,j}e_{i,k}) \\ &+ (s_{j,k}e_{i,j} - e_{i,k}e_{i,j}) = 0. \end{aligned}$$

The second equality is because $s_{i,j}s_{i,k} + s_{i,j}s_{j,k} = s_{j,k}s_{i,j} + s_{i,k}s_{i,j}$. For the third equality use Lemma 3.1, (6). For the fourth equality use lemma 3.1, (4). G -invariance of $\bar{\Omega}_n$ is evident. □

Let (E, ρ) be a finite dimensional representation of $B_n(m)$. By proposition 3.1, the connection

$$\rho(\bar{\Omega}_n) = \kappa \sum_{i < j} (\rho(s_{i,j}) - \rho(e_{i,j}))\omega_{i,j}$$

induce a flat connection on the bundle $Y_n \times_{S_n} E$, which is a linear bundle on X_n . Denote the resulted monodromy representation of $\pi_1(Y_n/S_n) \cong B_n$ as $\bar{\rho}$. The following theorem can be found in [Ma1].

Theorem 3.2 (Marin[Ma1]). *For generic κ , the monodromy representations $\bar{\rho}$ of B_n constructed above factor through $B_n(\tau, l)$, for $\tau = \frac{q^{1-m} - q^{m-1} + q^{-1} - q}{q^{-1} - q}$, $l = q^{m-1}$. Where $q = \exp \kappa \pi \sqrt{-1}$.*

4 Generalized Lawrence-Krammer Representations

The Lawrence-Krammer representations and their generalizations play a significant role in the theory of braid groups and Artin groups. See Krammer [Kr], Bigelow [Bi], Cohen and Wales [CW], Digne [Di], Paris [Pa] and Marin [Ma2] [Ma3]. Since this paper concentrate on infinitesimal level, we majorly refer to [Ma2] [Ma3].

Let V be a n -dimensional complex linear space. Let $G \subset U(V)$ be a complex reflection group. Let R be the set of reflections in G . We use notations in section 2.2.

The generalized LK representations of A_G of Marin are described by certain flat connections as follows. First, for every H_s , we have a closed 1-form ω_s on M_G as in section 2.2. Then let $V_G = \mathbb{C}\langle v_s \rangle_{s \in R}$ be a complex linear space with a basis indexed by R . For every pair of elements $s, u \in R$, define a nonnegative integer $\alpha(s, u) = \#\{r \in R | rur = s\}$. Chose a constant $m \in \mathbb{C}$. For any $s \in R$, define a linear operator $t_s \in GL(V_G)$ as follows:

$$t_s \cdot v_s = mv_s, \quad t_s \cdot v_u = v_{sus} - \alpha(s, u)v_s \quad \text{for } s \neq u.$$

Chose another constant $k \in \mathbb{C}$. Define a connection $\Omega_K = \sum_{s \in R} k \cdot t_s \omega_s$ on the trivial bundle $V_G \times M_G$.

Theorem and Definition (Marin[Ma2]) *The connection Ω_K is flat and G -invariant. So it induce a flat connection $\bar{\Omega}_K$ on the quotient bundle $M_G \times_G V_G$. The generalized LK representation for A_G is defined as the monodromy representation of $\bar{\Omega}_K$.*

We denote the generalized Krammer representation as $(V_G, \rho_{K,m})$. When (G, V) is the reflection group W_Γ of ADE type, they were first constructed in [CW] by Cohen, Wales and by Digne in [Di]. They are proved to factor through BMW algebras in [CGW1].

Theorem 4.1 (Marin [Ma1]). *The generalized Krammer representation $(V_G, \rho_{K,m})$ factor through the generalized BMW algebra $B_\Gamma(\tau, l)$ with $\tau = \frac{q^m - q^{-m} + q^{-1} - q}{q^{-1} - q}$ and $l = q^{-m}$. Where $q = e^{\kappa\pi\sqrt{-1}}$.*

For later convenience we change notations slightly. For $s \in R$, we define $p_s : V_G \rightarrow V_G$ by

$$p_s(v_s) = (1 - m_s)v_s, \quad p_s(v_u) = \alpha(s, u)v_s \quad \text{for } u \neq s.$$

We also define $\iota : G \rightarrow \text{Aut}(V_G)$ by $\iota(w)(v_s) = v_{wuw^{-1}}$. Then p_s is a projector to the complex line $\mathbb{C}v_s$. And Marin's flat connection Ω_K is written as $\sum_{s \in R} k \cdot (\iota(s) - p_s)\omega_s$.

A Further Generalization Let V be a n -dimensional complex linear space. Let $G \subset U(V)$ be a finite pseudo reflection group(not only complex reflection group). We define P, R, \mathcal{A} for G as in Section 1. For $s \in R$, denote the reflection hyperplane of s as H_s . Define $V_G = \mathbb{C}\langle v_i \rangle_{i \in P}$. Since $w(H_v)$ is another reflection hyperplane for any $w \in G$ and $v \in P$, there is an action of G on P which induce a representation $\iota : G \rightarrow \text{Aut}(V_G)$. Explicitly $w(i)$ is defined by $H_{w(i)} = w(H_v)$. For $i \in P$, let $p_i : V_G \rightarrow V_G$ be a projector to $\mathbb{C}v_i$ which

is written as:

$$p_i(v_i) = m_i v_i, \quad p_i(v_j) = \alpha_{i,j} v_i.$$

As in Section 1 let $\{\mu_s\}_{s \in R}$ be a set of nonzero constants such that: $\mu_{s_1} = \mu_{s_2}$ if s_1 is conjugate to s_2 in G . Define a function $i : R \rightarrow P$ such that $H_{i(s)}$ is the reflection hyperplane of s for any $s \in R$. Consider a connection Ω_{LK} on the trivial bundle $M_G \times V_G$ which have the form

$$\sum_{i \in P} \left(\sum_{s: i(s)=i} \mu_s \iota(s) - p_i \right) \omega_i.$$

Theorem 4.2. *The connection Ω_{LK} is flat and G -equivariant if and only if the the following conditions are satisfied:*

- (1) $m_i = m_j$ if there is $w \in G$ such that $\iota(w)(v_i) = v_j$.
- (2) $\alpha_{i,j} = \sum_{s: \iota(s)(v_j)=v_i} \mu_s$.

Proof. First we suppose Ω_{LK} is a flat, G -equivariant connection. By Proposition 2.2 we have,

$$\iota(w) \left(\sum_{s: i(s)=i} \mu_s \iota(s) - p_i \right) \iota(w)^{-1} = \sum_{s: i(s)=w(i)} \mu_s \iota(s) - p_{w(i)}.$$

By condition of the set $\{\mu_s\}_{s \in R}$, above identity is equivalent to $\iota(w)p_i \iota(w)^{-1} = p_{w(i)}$, which implies $m_i = m_{w(i)}$.

Let L be any codimension 2 edge of the arrangement \mathcal{A} . Let H_{i_1}, \dots, H_{i_N} be all the hyperplanes in \mathcal{A} containing L . By theorem 3.1, flatness of Ω_{LK} implies :

$$\left[\sum_{s: i(s)=i_a} \mu_s \iota(s) - p_{i_a}, \sum_{v=1}^N \left(\sum_{s: i(s)=i_v} \mu_s \iota(s) - p_{i_v} \right) \right] = 0. \quad (6)$$

for $1 \leq a \leq N$. Without losing generality suppose $a = 1$. It is equivalent to the following identity because by Proposition 2.1, the sum of those terms containing no p_i is zero.

$$\left[p_{i_1}, \sum_{v=1}^N \left(\sum_{s: i(s)=i_v} \mu_s \iota(s) - p_{i_v} \right) \right] + \left[\sum_{s: i(s)=i_1} \mu_s \iota(s), \sum_{v=1}^N p_{i_v} \right] = 0. \quad (7)$$

Now for those s such that $i(s) = i_1$ we have $\{s(i_1), \dots, s(i_N)\} = \{i_1, \dots, i_N\}$. So we have:

$$\begin{aligned} \left[\sum_{s: i(s)=i_1} \mu_s \iota(s), \sum_{v=1}^N p_{i_v} \right] &= \sum_{s: i(s)=i} \mu_s \sum_{v=1}^N (\iota(s)p_{i_v} - p_{i_v}\iota(s)) \\ &= \sum_{s: i(s)=i} \mu_s \sum_{v=1}^N (p_{s(i_v)}\iota(s) - p_{i_v}\iota(s)) \\ &= 0. \end{aligned} \quad (8)$$

So the identity (7) is equivalent to:

$$\begin{aligned}
[p_{i_1}, \sum_{v=1}^N (\sum_{s:i(s)=i_v} \mu_s \iota(s) - p_{i_v})] &= \sum_{v=2}^N \sum_{s:i(s)=i_v} [p_{i_1}, (\sum_{s:i(s)=i_v} \mu_s \iota(s) - p_{i_v})] \\
&= 0.
\end{aligned} \tag{9}$$

This is because $[p_{i_1}, \iota(s)] = 0$ if $i(s) = i_1$. After splitting the Lie bracket in equation (9), the sum of all those terms mapping to $\mathbb{C}v_{i_u}$ is $p_{i_u} p_{i_1} - \sum_{s:s(v_{i_1})=v_{i_u}} \mu_s \iota(s) p_{i_1}$. It must be 0. Chose s_0 such that $s_0(v_{i_1}) = v_{i_u}$ if there exist one, then we have $p_{i_u} p_{i_1} = \alpha_{i_u, i_1} \iota(s_0) p_{i_1}$. More over, for any s such that $s(v_{i_1}) = v_{i_u}$ we have $\iota(s) p_{i_1} = \iota(s_0) p_{i_1}$. Put these identities in equation $p_{i_u} p_{i_1} - \sum_{s:s(v_{i_1})=v_{i_u}} \mu_s \iota(s) p_{i_1} = 0$, we get

$$(\alpha_{i_u, i_1} - \sum_{s:s(v_{i_1})=v_{i_u}} \mu_s) \iota(s_0) p_{i_1} = 0.$$

So we have $\alpha_{i_u, i_1} = \sum_{s:s(v_{i_1})=v_{i_u}} \mu_s$. If there don't exist such s_0 , we can prove $\alpha_{i_u, i_1} = 0$ similarly.

Now suppose conditions (1) and (2) are satisfied, by the same arguments we only need to prove above equation (9) to show Ω_{LK} is flat. The conditions (2) implies

$$p_i p_j = \sum_{s:s(j)=i} \mu_s \iota(s) p_j, \text{ for any } i \neq j. \tag{10}$$

It also implies

$$p_i p_j = \sum_{s:s(j)=i} \mu_s p_i \iota(s), \text{ for any } i \neq j. \tag{11}$$

since $\iota(s) p_j = \iota(s) p_j \iota(s)^{-1} \iota(s) = p_i \iota(s)$ for those s such that $s(j) = i$. Now the right hand side of equation (9) can be written as

$$\sum_{v=2}^N (p_{i_1} p_{i_v} - \sum_{s:i(s)=i_1} \mu_s p_{i_1} \iota(s)) + \sum_{v=2}^N (p_{i_v} p_{i_1} - \sum_{s:i(s)=i_v} \mu_s \iota(s) p_{i_v}).$$

So the equation (6) is true and it implies flatness of Ω_{LK} by Theorem 3.1. G-equivariance of the connection is easy to see. □

Remark In the connection Ω_{LK} if make $\mu_s = \kappa$ for all s and $m_i = m$ for all i then we obtain Marin's connection. Above theorem produces flat, G-equivariant connections with more parameters. It also explains the number $\alpha_{i,j}$ in Marin's construction.

Definition 4.1 (Generalized LK representations for general complex braid groups). *Following notations introduced above. Since Ω_{LK} is G-invariant, it induces a flat connection $\bar{\Omega}_{LK}$ on the quotient bundle $M_G \times_G V_G$. The generalized Lawrence-Krammer representation of the braid group A_G is defined as the monodromy representation of $\bar{\Omega}_{LK}$.*

Suppose P_1, P_2, \dots, P_{N_G} are all equivalent classes of P under the equivalence relation ' \sim '. The following lemma is essentially from [Ma2].

Lemma 4.1. *For any $1 \leq N \leq N_G$, the subspace $V_N = \oplus_{i \in P_N} \mathbb{C}v_i \subset V_G$ is a subrepresentation.*

Proof. We only need to observe that $w(i) \sim i$ for any $i \in P$ and $w \in G$, and $p_i(v_j) = 0$ if $i \not\sim j$.

5 Basic Properties about $B_G(\Upsilon)$

Suppose $G \subset U(V)$ is a finite pseudo reflection group. Define $B_G(\Upsilon)$ as in Definition 1.1. When G is a complex reflection group, there is a bijection from R to \mathcal{A} : $s \mapsto H_s$. So we can use R as the indices set P of reflection hyperplanes. In these cases, for $s_1, s_2 \in R$, $R(s_1, s_2) = \{s \in R | s(H_{s_2}) = H_{s_1}\} = \{s \in R | ss_2s = s_1\}$.

Theorem 5.1. *When G is finite then $B_G(\Upsilon)$ is a finite dimensional algebra. Moreover, the map $w \mapsto T_w$ for $w \in G$ induce an injection $j: \mathbb{C}G \rightarrow B_G(\Upsilon)$.*

Proof. First by using relation (3), we can identify any word made from the set $\{w \in G\} \amalg \{e_i\}_{i \in P}$ with a word of the form $T_w e_{i_1} e_{i_2} \cdots e_{i_k}$ where $w \in G$. We define the e-length of such a word as k . In this word if two neighboring e_{i_v} and $e_{i_{v+1}}$ don't commute with each other, then for $e_{i_v} e_{i_{v+1}}$, condition in (5) of Definition 1.1 is satisfied as shown by the next lemma.

Lemma 5.1. *If two pseudo reflection s_1 and s_2 don't commute with each other, suppose the reflection hyperplane of $s_1(s_2)$ is $H_{i_1}(H_{i_2})$, then $\{i_1, i_2\} \subsetneq \{k \in P | H_k \supseteq H_{i_1} \cap H_{i_2}\}$.*

Proof. We suppose $\{i_1, i_2\} = \{k \in P | H_k \supseteq H_{i_1} \cap H_{i_2}\}$. Let $L = H_{i_1} \cap H_{i_2}$, and \langle, \rangle being a G -invariant inner product on V . Chose $v_k \in H_{i_k}$ such that $v_k \perp L$ according to \langle, \rangle for $k = 1, 2$. Suppose $\{v_3, \dots, v_N\}$ is a basis of L , then $\{v_1, v_2, \dots, v_N\}$ is a basis of V . Now since $s_1(H_{i_2})$ is another reflection hyperplane containing L and $s_1(H_{i_2}) \neq H_{i_1}$, so we have $s_1(H_{i_2}) = H_{i_2}$, which implies s_1 can be presented as a diagonal matrix according to the basis $\{v_1, \dots, v_N\}$. Similarly s_2 can be presented by a diagonal matrix according to the same basis. So $s_1 s_2 = s_2 s_1$ which is a contradiction. \square

The first statement of theorem 5.1 follows from the next lemma.

Lemma 5.2. *The algebra $B_G(\Upsilon)$ is spanned by the set*

$$\{T_w e_{i_1} \cdots e_{i_M} | w \in G; e_{i_u} e_{i_v} = e_{i_v} e_{i_u} \text{ and } i_v \neq i_u \text{ if } u \neq v; M \geq 0\}$$

Proof. Let A be the subspace in $B_G(\Upsilon)$ spanned by elements listed in the lemma. We only need to prove any word $T_w e_{i_1} e_{i_2} \cdots e_{i_k}$ represents an element in A . We do it by induction on e-length of such words. First this is true if $K = 1$. Suppose it is true for $K \leq M$. Now consider a word $x = w e_{i_1} \cdots e_{i_{M+1}}$. If there are two neighboring $e_{i_v}, e_{i_{v+1}}$ don't commute,

then Lemma 5.1 enable us to apply (5) or (6) in Definition 1.1 to identify x with a linear sum of words whose e-tail length are smaller than $M + 1$. Suppose all e_{i_v} 's in x commute with each other, if there are v_1, v_2 such that $i_{v_1} = i_{v_2}$, we use permutations between e_{i_v} 's to identify x with a word $y = we_{j_1} \cdots e_{j_{M+1}}$ such that $j_1 = j_2$. So $x = y = m_{j_1} we_{j_2} \cdots e_{j_{M+1}}$ by relation (2) of Definition 1.1. If all e_{i_v} 's commute and all i_v 's are different then $x \in A$, and induction is completed. For the second statement of theorem 5.1, it isn't hard to see the following map

$$T_w \mapsto w, \text{ for } w \in G; e_i \mapsto 0, \text{ for } i \in P$$

extends to a surjection $\pi : B_G(\Upsilon) \rightarrow \mathbb{C}G$, and $\pi \circ j = \text{id}$. So j is injective. \square

This completes the proof of Theorem 5.1. \square

By Theorem 5.1, $\mathbb{C}G$ is naturally embedded in $B_G(\Upsilon)$. For saving notations from now on we always think $\mathbb{C}G$ to be included in $B_G(\Upsilon)$, and denote T_w simply as w . The next lemma reduce one parameter in $B_G(\Upsilon)$.

Lemma 5.3. *For $\lambda \in \mathbb{C}^\times$, Let $\mu'_s = \lambda\mu_s$ for $s \in R$, and Let $m'_i = \lambda m_i$ for $i \in P$. Let $\Upsilon' = \{\mu'_s, m'_i\}_{s \in R, i \in P}$, then $B_G(\Upsilon') \cong B_G(\Upsilon)$.*

Proof. Denote the generators of $B_G(\Upsilon')$ appeared in Definition 1.1 as S_i 's and E_i 's. Then

$$s_i \mapsto S_i, e_i \mapsto \lambda E_i \text{ for } i \in P$$

extend to an isomorphism from $B_G(\Upsilon)$ to $B_G(\Upsilon')$. \square

The following lemma can be found in [Ma2].

Lemma 5.4 (Marin). *For two different hyperplane $H_i, H_j \in \mathcal{A}$, If $s \in R$ satisfies $s(H_j) = H_i$, then s fix all points in $H_i \cap H_j$. So, $R(i, j) = \{s \in R | s(H_j) = H_i; H_s \supseteq H_i \cap H_j\}$.*

Proof. Let \langle, \rangle be a G -invariant inner product on V . Let ϵ be the exceptional eigenvalue of s , and let u be an eigenvalue of s with eigenvalue ϵ . Let u_i, u_j be some nonzero vectors perpendicular to H_i, H_j respectively. Then $u \perp H_s$. The action of s on V can be written as $s(v) = v - (1 - \epsilon) \frac{\langle v, u \rangle}{\langle u, u \rangle} u$. Now $s(H_j) = H_i$ implies $s(u_j) = u_j - (1 - \epsilon) \frac{\langle u_j, u \rangle}{\langle u, u \rangle} u = \lambda u_i$ for some $\lambda \neq 0$. Denote $(1 - \epsilon) \frac{\langle u_j, u \rangle}{\langle u, u \rangle}$ as κ . The condition that H_i is different from H_j implies $\kappa \neq 0$. So we have $u = \frac{1}{\kappa}(u_j - \lambda u_i) \perp H_i \cap H_j$, and s fix all points in $H_i \cap H_j$. \square

There exists a natural anti-involution on $B_G(\Upsilon)$ which may be used to construct a cellular structure as follows.

Lemma 5.5. *The following map extends to an anti-involution $*$ of $B_G(\Upsilon)$*

$$w \mapsto w^{-1} \text{ for } w \in G \subset B_G(\Upsilon), e_i \mapsto e_i \text{ for all } i \in P$$

if $\mu_s = \mu_{s^{-1}}$ for any $s \in R$.

Proof. We only need to certify * keep all relations in Definition 1.1. As an example for relation (5), on the one hand $*(e_i e_j) = e_j e_i$, on the other hand $*[(\sum_{s \in R(i,j)} \mu_s s) e_j] = e_j (\sum_{s \in R(i,j)} \mu_s s^{-1}) = (\sum_{s \in R(i,j)} \mu_s s^{-1}) e_i = (\sum_{s \in R(j,i)} \mu_{s^{-1}} s) e_i = (\sum_{s \in R(j,i)} \mu_s s) e_i$. \square

Flat Connections

Define a formal connection $\bar{\Omega}_G = \kappa \sum_{i \in P} (\sum_{s: i(s)=i} \mu_s s - e_i) \omega_i$.

Suppose $\rho : B_G(Y) \rightarrow \text{End}(E)$ is a finite dimensional representation. On the vector bundle $M_G \times E$, we define a connection $\rho(\bar{\Omega}_G) = \kappa \sum_{i \in P} (\sum_{s: i(s)=i} \mu_s \rho(s) - \rho(e_i)) \omega_i$ where $\kappa \in \mathbb{C}^\times$. Let G acts on $M_G \times E$ as $w \cdot (x, v) = (wx, \rho(w)v)$ for $w \in G$ and $(x, v) \in M_G \times E$.

Proposition 5.1. *The connection $\rho(\bar{\Omega}_G)$ and $\bar{\Omega}_G$ are flat and G -invariant.*

Proof. It is enough to deal with the case $\kappa = 1$. By Proposition 2.2, to show the G -invariance we only need to prove

$$\sum_{s: i(s)=i} \mu_s \rho(w) \rho(s) \rho(w)^{-1} - \rho(w) \rho(e_i) \rho(w)^{-1} = \sum_{s: i(s)=w(i)} \mu_s \rho(s) - \rho(e_{w(i)}) \quad (12)$$

for any $w \in G$. By (3) of Definition 1.1, we have $\rho(w) \rho(e_i) \rho(w)^{-1} = \rho(w e_i w^{-1}) = \rho(e_{w(i)})$. We also have $\{w s w^{-1} | i(s) = i\} = \{s | i(s) = w(i)\}$ and $\mu_s = \mu_{w s w^{-1}}$, so identity (12) follows.

Let L be any codimension 2 edge for the arrangement \mathcal{A} , and let H_{i_1}, \dots, H_{i_N} be all the hyperplanes in \mathcal{A} containing L . By Theorem 3.1, to prove $\rho(\bar{\Omega}_G)$ is flat we need to show for any u

$$[\sum_{s: i(s)=i_u} \mu_s \rho(s) - \rho(e_{i_u}), \sum_{v=1}^N (\sum_{s: i(s)=i_v} \mu_s \rho(s) - \rho(e_{i_v}))] = 0. \quad (13)$$

Now remember the connection $\kappa \sum_{i \in P} (\sum_{s: i(s)=i} \mu_s \rho(s) \omega_i)$ is flat by proposition 2.1, so (13) is equivalent to

$$[\sum_{s: i(s)=i_u} \mu_s \rho(s), \sum_{v=1}^N \rho(e_{i_v})] + [\rho(e_{i_u}), \sum_{v=1}^N (\sum_{s: i(s)=i_v} \mu_s \rho(s) - \rho(e_{i_v}))] = 0 \quad (14)$$

Because for any s such that $i(s) = i_u$, there is $\{s(H_{i_1}), \dots, s(H_{i_N})\} = \{H_{i_1}, \dots, H_{i_N}\}$, so

$$\begin{aligned} \rho(s) \sum_{v=1}^N \rho(e_{i_v}) - \sum_{v=1}^N \rho(e_{i_v}) \rho(s) &= (\rho(s) \sum_{v=1}^N \rho(e_{i_v}) \rho(s)^{-1} - \sum_{v=1}^N \rho(e_{i_v})) \rho(s) \\ &= (\sum_{v=1}^N \rho(e_{s(i_v)}) - \sum_{i=1}^N \rho(e_{i_v})) = 0 \end{aligned} \quad (15)$$

So (14) is equivalent to

$$[\rho(e_{i_u}), \sum_{v=1}^N (\sum_{s: i(s)=i_v} \mu_s \rho(s) - \rho(e_{i_v}))] = 0. \quad (16)$$

We define $I_1 = \{1 \leq v \leq N | s(i_v) = i_u, \text{ for some } s \in \mathcal{R}\}$, and $I_2 = \{1 \leq v \leq N | s(i_v) \neq i_u, \text{ for any } s \in \mathcal{R}\}$. There is $\{1, 2, \dots, N\} = I_1 \sqcup I_2$.

$$\begin{aligned}
[\rho(e_{i_u}), \sum_{v=1}^N (\sum_{s: i(s)=i_v} \mu_s \rho(s) - \rho(e_{i_v}))] &= - \sum_{v \in I_1} (\rho(e_{i_u} e_{i_v}) - \sum_{s: s(i_v)=i_u} \mu_s \rho(e_{i_u}) \rho(s)) \\
&\quad + \sum_{v \in I_1} (\rho(e_{i_v} e_{i_u}) - \sum_{s: s(i_u)=i_v} \mu_s \rho(s) \rho(e_{i_u})) \\
&\quad + \sum_{v \in I_2} (\rho(e_{i_u} e_{i_v}) - \rho(e_{i_v} e_{i_u})) \\
&= 0.
\end{aligned} \tag{17}$$

Where we used relation (5),(6) in Definition 1.1.

Flatness of $\bar{\Omega}_G$ can be proved similarly. □

Theorem 5.2. *Using notations in section 4. The map $w \mapsto \iota(w)$, $e_i \mapsto p_i$ extends to a representation $B_G(\Upsilon) \rightarrow \text{End}(V_G)$. So from $B_G(\Upsilon)$ we can obtain the generalized Lawrence-Krammer representation.*

Proof. We only need to certify that $\iota(w)$'s and p_i 's satisfy those relations in Definition 1.1. Relation (0) is evident. Relation (1) and (1)' are because p_i is a projector to $\mathbb{C}v_i$. Relation (3) is by definition of p_i and the fact $\alpha_{i,j} = \alpha_{w(i),w(j)}$. When $R(i,j) = \emptyset$, by definition we have $p_i p_j = p_j p_i = 0$ so relation (4), (6) follows. For any i, j, k , we have

$$\begin{aligned}
p_i p_j (v_k) &= \alpha_{j,k} p_i (v_j) = \alpha_{j,k} \alpha_{i,j} v_i \text{ and} \\
(\sum_{s \in R(i,j)} \mu_s \iota(s)) p_j (v_k) &= \alpha_{j,k} (\sum_{s \in R(i,j)} \mu_s \iota(s)) (v_j) = \alpha_{j,k} (\sum_{s \in R(i,j)} \mu_s) v_i = \alpha_{j,k} \alpha_{i,j} v_i,
\end{aligned}$$

so relation (5) is certified. □

Suppose Γ is a finite type simply laced Dynkin diagram, denote the associated Coxeter group and Artin group as W_Γ , A_Γ respectively. Suppose W_Γ is realized as a reflection group in $U(V)$. In this case the data Υ consists of two constants m, μ since all reflections of W_Γ lie in the same conjugacy class. Suppose $\mu \neq 0$, so by Lemma 5.3 we can set $\mu = 1$. Thus we denote the algebra of Definition 1.1 for W_Γ as $B_{W_\Gamma}(m)$. Set $M_\Gamma = V \setminus \cup_{i \in P} H_i$. Let (E, ρ) be a finite dimensional representation of $B_{W_\Gamma}(m)$. By proposition 5.1, the connection

$$\rho(\bar{\Omega}_\Gamma) = \kappa \sum_{i \in P} (\rho(s_i) - \rho(e_i)) \omega_i$$

induces a flat connection on the bundle $M_\Gamma \times_{W_\Gamma} E$, which is a linear bundle on M_Γ/W_Γ . Denote the resulted monodromy representation of $\pi_1(M_\Gamma/W_\Gamma) \cong A_\Gamma$ as $\bar{\rho}$. We have

Theorem 5.3. *If $m \notin \mathbb{Z}$ or $m \in \mathbb{Z}$ with $m > 3$, the monodromy representations $\bar{\rho}$ of A_Γ constructed above factor through the simply laced BMW algebra $B_\Gamma(\tau, l)$, for $\tau = \frac{q^{1-m} - q^{m-1} + q^{-1} - q}{q^{-1} - q}$, $l = q^{m-1}$. Where $q = \exp \kappa \pi \sqrt{-1}$.*

Proof. Suppose Σ is the set of nodes of Γ , $\{\sigma_i\}_{i \in \Sigma}$ is the set of generators of A_Γ in a canonical presentation. For $i \in \Sigma$, set

$$X_i = \bar{\rho}(\sigma_i), \quad E_i = \frac{q^{-1} - q}{l} (\bar{\rho}(\sigma_i)^2 + (q^{-1} - q) \bar{\rho}(\sigma_i) - 1).$$

We need to show $\{X_i, E_i\}_{i \in \Sigma}$ satisfies relations of $B_\Gamma(\tau, l)$ in table 2. The proof is completely similar to Theorem 3.2, so we content with giving a sketch. Denote the number of nodes in Γ as $n(\Gamma)$. We only consider the cases when Γ is irreducible. When $n(\Gamma) = 1$ or 2, the Artin group A_Γ is braid group B_2, B_3 respectively. So the statement follows from Theorem 3.2. Suppose $n(\Gamma) \geq 3$. The fact that Γ is simply laced enable us to reduce the statement of these cases to cases when $n(\Gamma) = 1$ or 2, by using Theorem 2.1. Suppose $i, j \in \Sigma$ and $i \sim j$. Then the parabolic subgroup $W_{i,j}$ of W_Γ generated by s_i, s_j is isomorphic to the symmetric group S_3 . By applying Theorem 2.1 to $W_{i,j}$, we prove that X_i, X_j, E_i, E_j satisfies relations in table 2. The cases when $i \approx j$ can be proved similarly. \square

6 Cases of Dihedral Groups

Dimension and Basis Denote the dihedral group of type $I_2(m)$ as G_m . The arrangement of its reflection hyperplanes can be explained with the following Figure 2.

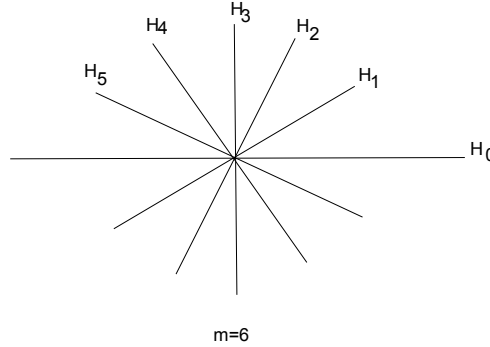


Figure 2: The arrangement of G_6

There are m lines(hyperplanes) passing the origin. The angle between every two neighboring lines is π/m . Suppose the x -axis is one of the reflection lines and denote it as H_0 , we denote these lines by H_0, H_1, \dots, H_{m-1} in anticlockwise order as shown in above graph. Denote the reflection by H_i as s_i . The set of reflections in G_m is $R = \{s_i\}_{0 \leq i \leq m-1}$. Denote the rotation of $2j\pi/m$ in anticlockwise order as r_j . It is well known that G_m is generated by s_0, s_1 with the following presentation

$$\langle s_0, s_1, \mid (s_0 s_1)^m = 1, s_0^2 = s_1^2 = 1 \rangle.$$

As a set $G_m = \{s_i, r_i\}_{0 \leq i \leq m-1}$. Under this presentation, s_i can be determined inductively in the following way. $s_1 = s_1, s_2 = s_1 s_0 s_1$ and $s_i = s_{i-1} s_{i-2} s_{i-1}$. By $[s_i s_j \dots]_k$ we denote

the length k word starting with $s_i s_j$, in which s_i and s_j appear alternatively. The word $[\cdots s_0 s_1]_k$ is defined similarly. Then $s_i = [s_1 s_0 \cdots]_{2i-1}$, where $s_m = [s_1 s_0 \cdots]_{2m-1} = s_0$.

As for the algebra $B_{G_m}(\Upsilon)$, we choose the index set $P_{G_m} = \{0, 1, 2, \dots, m-1\}$. $B_{G_m}(\Upsilon)$ is generated by $\{s_i, e_i\}_{i \in P_{G_m}}$. The data Υ is $\{\mu_i, \tau_i\}_{i \in P_{G_m}}$. For later convenience, for $k \in \mathbb{Z}$, we define $s_k, H_k, e_k, \mu_k, \tau_k$ as $s_{[k]}, H_{[k]}, e_{[k]}, \mu_{[k]}, \tau_{[k]}$ respectively. Where $[k]$ is the unique number in $\{0, 1, \dots, m-1\}$ such that $k \equiv [k] \pmod{m\mathbb{Z}}$.

The structure of $B_{G_m}(\Upsilon)$ when m is odd is quite different from cases when m is even.

- When $m = 2k + 1$ is odd, we have $i \sim j$ for any $i, j \in P_{G_m}$. Which implies $\mu_i = \mu_j$ and $\tau_i = \tau_j$ for any $i, j \in P_{G_m}$. So the data Υ consists of $\{\tau_0, \mu_0\}$ essentially. We always suppose $\mu_0 \neq 0$, and by Lemma 5.3 we can suppose $\mu_0 = 1$. When $i + j \in 2\mathbb{Z}$, $R(i, j) = \{s_{\frac{i+j}{2}}\}$; when $i + j + 1 \in 2\mathbb{Z}$, $R(i, j) = \{s_{\frac{i+j+m}{2}}\}$.
- When $m = 2k$ is even, $i \sim j$ if and only if $i + j \in 2\mathbb{Z}$. Which implies $\mu_i = \mu_j$ and $\tau_i = \tau_j$ if $i + j \in 2\mathbb{Z}$. So the data Υ consists of $\{\mu_0, \mu_1, \tau_0, \tau_1\}$ essentially. When $i + j \in 2\mathbb{Z}$, $R(i, j) = \{s_{\frac{i+j}{2}}, s_{\frac{i+j+m}{2}}\}$; when $i + j + 1 \in 2\mathbb{Z}$, $R(i, j) = \emptyset$.

It is easy to see when $m = 2k$ is even, the relation (1)' in Definition 5.1 for $B_{G_m}(\Upsilon)$ is equivalent to: $s_{i+k} e_i = e_i s_{i+k} = e_i$ for any i .

Theorem 6.1. (1) When m is odd, the algebra $B_{G_m}(\Upsilon)$ has dimension $2m + m^2$, and has the set $\{s_i, r_i\}_{0 \leq i \leq m-1} \cup \{s_i e_j\}_{0 \leq i, j \leq m-1}$ as a basis.

(2) When $m = 2k$ is even, the algebra $B_{G_m}(\Upsilon)$ has dimension $2m + \frac{m^2}{2}$, and has the set $\{s_i, r_i\}_{0 \leq i \leq m-1} \cup \{s_i e_{2j}\}_{0 \leq i, j \leq k-1} \cup \{s_i e_{2j+1}\}_{0 \leq i, j \leq k-1}$ as a basis.

When m is odd, let A_m be the vector space spanned by some generators $\{S_i, R_i\}_{0 \leq i \leq m-1} \cup \{T_{i,j}\}_{0 \leq i, j \leq m-1}$. For convenience for any $i, j \in \mathbb{Z}$, let $S_i = S_{[i]}$, $R_i = R_{[i]}$ and $T_{i,j} = T_{[i],[j]}$. Define a product on A_m by the following relations (1), (2), (3).

$$(1) \quad S_i S_j = R_{i-j}, R_i R_j = R_{i+j}, S_i R_j = S_{i-j}, R_j S_i = S_{i+j}.$$

$$(2) \quad S_l T_{i,j} = T_{l-i+j,j}, T_{i,j} S_l = T_{i-j+l, 2l-j}, R_l T_{i,j} = T_{i+l,j}, T_{i,j} R_l = T_{i-l,j-2l}.$$

(3)

$$T_{i,j} T_{p,q} = \begin{cases} \tau_q T_{i-p+q,q} & \text{when } 2p-j-q \in m\mathbb{Z}; \\ T_{v_{i,j,p,q},q} & \text{when } 2p-j-q \notin m\mathbb{Z}, \text{ and } [2p-j] + q \in 2\mathbb{Z}; \\ T_{u_{i,j,p,q},q} & \text{when } 2p-j-q \notin m\mathbb{Z}, \text{ and } [2p-j] + q \notin 2\mathbb{Z}. \end{cases}$$

Where $v_{i,j,p,q} = (2i + [2p-j] + q - 2p)/2$, $u_{i,j,p,q} = (2i + [2p-j] + q + m - 2p)/2$.

Lemma 6.1. Above product makes A_m into an associative algebra.

Proof. In fact above identities are obtained by "looking $(S_i, R_i, T_{i,j})$ as $(s_i, r_i, s_i e_j)$ ". We have an indirect proof as follows. Denote the m -dimensional representation of $B_G(\Upsilon)$ defined in Theorem 5.2 as ρ_{LK} , the irreducible representations of $B_G(\Upsilon)$ induced by the surjection

$\pi : B_G(\Upsilon) \rightarrow \mathbb{C}G$ as ρ_1, \dots, ρ_l . Denote the parameter space of all Υ 's as Λ . By similar argument with the proof of (2) of Proposition 7.1, we can show there is a dense open subset \mathcal{D} of Λ such that if $\Upsilon \in \mathcal{D}$ then the related representation ρ_{LK} is irreducible. In these cases by Wedderburn-Artin Theorem we have $\dim B_G(\Upsilon) \geq \sum_i (\dim \rho_i)^2 + (\dim \rho_{LK})^2 = 2m + m^2$. Since the set $\{s_i, r_i\}_{0 \leq i \leq m-1} \cup \{s_i e_j\}_{0 \leq i, j \leq m-1}$ always spans $B_G(\Upsilon)$, so we know when $\Upsilon \in \mathcal{D}$, this set is a basis of $B_G(\Upsilon)$. Thus the product of A_m is associative if $\Upsilon \in \mathcal{D}$. So the product is associative for all Υ . \square

Proof of theorem 6.1 When m is odd, denote the algebra above as $A_m(\Upsilon)$. we define a map ϕ as: $\phi(S_i) = s_i$, $\phi(R_i) = r_i$, for $0 \leq i \leq m-1$; $\phi(T_{i,j}) = s_i e_j$ for $0 \leq i, j \leq m-1$. It is easy to see ϕ extends to a morphism ϕ from $A_m(\Upsilon)$ to $B_{G_m}(\Upsilon)$. Inversely the map ψ : $\psi(s_i) = S_i$, $\psi(e_i) = T_{i,i}$ for $0 \leq i \leq m-1$ extends to a morphism ψ from $B_{G_m}(\Upsilon)$ to $A_m(\Upsilon)$. Since $\psi\phi = \text{id}$ and $\phi\psi = \text{id}$, we know $B_{G_m}(\Upsilon)$ is isomorphic to $A_m(\Upsilon)$ and statement (1) follows. The statement (2) can be proved similarly by constructing an actual algebra with dimension $2m + \frac{m^2}{2}$ and prove it is isomorphic to $B_{G_m}(\Upsilon)$.

Cellular Structures When m is Odd Suppose $m = 2k + 1$. let $(\Lambda, M, C, *)$ be the cellular structure of $\mathbb{C}G_m$. The algebra $B_{G_m}(\Upsilon)$ has a cellular structure $(\bar{\Lambda}, \bar{M}, \bar{C}, *)$ as follows.

- $\bar{\Lambda} = \Lambda \sqcup \{\lambda_{LK}\}$. We keep the original partial order in Λ and for any $\lambda \in \Lambda$, let $\lambda_{LK} \prec \lambda$.
- For $\lambda \in \Lambda$, set $\bar{M}(\lambda) = M(\lambda)$ and $\bar{M}(\lambda_{LK}) = \{0, 1, \dots, m-1\}$.
- For $\lambda \in \Lambda$, and $S, T \in \bar{M}(\lambda)$ set $\bar{C}_{S,T}^\lambda = C_{S,T}^\lambda$. For $i, j \in \bar{M}(\lambda_{LK})$, set $\bar{C}_{i,j}^{\lambda_{LK}} = s_{\frac{i+j}{2}} e_j$ if $i+j \in 2\mathbb{Z}$, and $\bar{C}_{i,j}^{\lambda_{LK}} = s_{\frac{i+j+m}{2}} e_j$ if $i+j \notin 2\mathbb{Z}$.
- Define $*$ to be the involution in Lemma 5.5.

Theorem 6.2. *Above $(\bar{\Lambda}, \bar{M}, \bar{C}, *)$ defines a cellular structure for $B_{G_m}(\Upsilon)$.*

Proof. Recall the definition of cellular algebras in section 2. (C1) follows by Theorem 6.1. (C2) is because $*(s_{\frac{i+j}{2}} e_j) = e_j s_{\frac{i+j}{2}} = s_{\frac{i+j}{2}} e_i$. (C3) is by the following computation. $s_l(\bar{C}_{i,j}^{\lambda_{LK}}) = \bar{C}_{2l-i,j}^{\lambda_{LK}}$ for any l, i, j ; $e_l(\bar{C}_{i,j}^{\lambda_{LK}}) = \bar{C}_{l,j}^{\lambda_{LK}}$ if $l \neq i$; $e_i(\bar{C}_{i,j}^{\lambda_{LK}}) = \tau_0 \bar{C}_{i,j}^{\lambda_{LK}}$.

Remark 6.1. *From above proof we see the cellular representation corresponding to λ_{LK} is the infinitesimal LK representation of Marin.*

Cellular Structures When m is Even Suppose $m = 2k$. Still denote the cellular structure of $\mathbb{C}G_m$ as $(\Lambda, M, C, *)$. The algebra $B_{G_m}(\Upsilon)$ has a cellular structure $(\bar{\Lambda}, \bar{M}, \bar{C}, *)$ as follows.

- $\bar{\Lambda} = \Lambda \cup \{\lambda_{LK^0}, \lambda_{LK^1}\}$. We keep the partial order in Λ , and let $LK^i \prec \lambda$ for any $\lambda \in \Lambda$ and any i .

- For $\lambda \in \Lambda$, $\bar{M}(\lambda) = M(\lambda)$. $\bar{M}(\lambda_{LK^0}) = \{0, 2, \dots, 2k-2\}$. $\bar{M}(\lambda_{LK^1}) = \{1, 3, \dots, 2k-1\}$.
- For $\lambda \in \Lambda$, $S, T \in \bar{M}(\lambda)$, let $\bar{C}_{S,T}^\lambda = C_{S,T}^\lambda$.
- $\bar{C}_{2i,2j}^{\lambda_{LK^0}} = \frac{1}{2}(s_{i+j} + s_{i+j+k})e_{2j}$, $\bar{C}_{2i+1,2j+1}^{\lambda_{LK^1}} = \frac{1}{2}(s_{i+j+1} + s_{i+j+k+1})e_{2j+1}$.
- Let $*$ be the involution in lemma 5.5.

Theorem 6.3. *Above $(\bar{\Lambda}, \bar{M}, \bar{C}, *)$ defines a cellular structure for $B_{G_m}(\Upsilon)$.*

Proof. (C1) follows from Theorem 6.1. (C2) is certified similarly. (C3) follows from the following computation. $s_l(\bar{C}_{2i,2j}^{\lambda_{LK^0}}) = \bar{C}_{2(l-i),2j}^{\lambda_{LK^0}}$; $e_l(\bar{C}_{2i,2j}^{\lambda_{LK^0}}) = 0$ if l is odd; $e_{2i}(\bar{C}_{2i,2j}^{\lambda_{LK^0}}) = \tau_0 \bar{C}_{2i,2j}^{\lambda_{LK^0}}$; $e_l(\bar{C}_{2i,2j}^{\lambda_{LK^0}}) = (\mu_{p+i} + \mu_{p+i+k})\bar{C}_{l,2j}^{\lambda_{LK^0}}$ if $l = 2p$ is even and $l \neq 2i$.
 $s_l(\bar{C}_{2i+1,2j+1}^{\lambda_{LK^1}}) = \bar{C}_{2(l-i-1)+1,2j+1}^{\lambda_{LK^1}}$; $e_l(\bar{C}_{2i+1,2j+1}^{\lambda_{LK^1}}) = 0$ if l is even; $e_{2i+1}(\bar{C}_{2i+1,2j+1}^{\lambda_{LK^1}}) = \tau_1 \bar{C}_{2i+1,2j+1}^{\lambda_{LK^1}}$; $e_l(\bar{C}_{2i+1,2j+1}^{\lambda_{LK^1}}) = (\mu_{p+i+1} + \mu_{p+i+1+k})\bar{C}_{2p+1,2j+1}^{\lambda_{LK^1}}$ if $l = 2p+1$ is odd and $l \neq 2i+1$. \square

Remark 6.2. *The two representations corresponding to λ_{LK^0} , λ_{LK^1} are components of the infinitesimal LK representations of Marin as in Theorem 5.2.*

7 H_3 Type

The Coxeter group G_{H_3} of type H_3 is the symmetric group of a regular dodecahedron (or a regular icosahedron). It is generated by s_0, s_1, s_2 with relations:

- | | |
|--------------------------------|--|
| a) $s_i^2 = 1$ for all i 's. | b) $s_0 s_1 s_0 s_1 s_1 = s_1 s_0 s_1 s_0 s_1$. |
| c) $s_0 s_2 = s_2 s_0$. | d) $s_1 s_2 s_1 = s_2 s_1 s_2$. |

We have: $|G_{H_3}| = 120$, $|R| = 15$.

The group G_{H_3} has a nontrivial center element $c = (s_2 s_0 s_1)^5$ which is also the longest element. Denote the other 12 reflections arbitrarily as s_3, \dots, s_{14} so $R = \{s_i\}_{0 \leq i \leq 14}$. Denote the reflection hyperplane of s_i as H_i , naturally set the index set of reflection hyperplanes as $P = \{0, 1, \dots, 14\}$.

In the following Figure 3, the dotted lines show the intersection of three reflection hyperplanes with the front surface of the dodecahedron. For $s, s' \in R$, we say s is perpendicular to s' and denote $s \perp s'$ if $ss' = s's$. From Figure 3 we see directly that $' \perp '$ is a equivalent relation in R (the proof of this fact is only simple but lengthy computations). According to it R is decomposed into 5 equivalent classes $\mathcal{R} = \{R_1, \dots, R_5\}$, each class consists of 3 elements. Let $w_0 = s_1 s_2 s_1 s_0 s_1 s_0 s_1$. A typical equivalent class is $\{s_0, s_2, s_{i_0}\}$ where $s_{i_0} = w_0^{-1} s_0 w_0$. Any way we suppose $R_\alpha = \{i_\alpha, j_\alpha, k_\alpha\}$ for $1 \leq \alpha \leq 5$ and let $R_1 = \{s_0, s_2, s_{i_0}\}$. The conjugating action of G_{H_3} on R induces an action of the same group on \mathcal{R} , because $s_i \perp s_j$ implies $ws_i w^{-1} \perp ws_j w^{-1}$. Since G_{H_3} acts on R transitively by conjugation, the action of G_{H_3} on \mathcal{R} is also transitive.

It isn't hard to see that, $|R(i, j)| = 0$ if $s_i \perp s_j$ and $|R(s_i, s_j)| = 1$ otherwise.

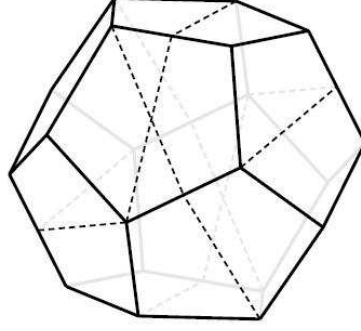


Figure 3: Regular dodecahedron

Now consider the algebra $B_{G_{H_3}}(\Upsilon)$. Since all elements of R lie in the same conjugacy class, so all τ_i equal and all μ_i equal. We denote them as τ and μ respectively. Suppose $\mu \neq 0$, in these cases we can set $\mu = 1$ by lemma 5.3. We have

Lemma 7.1.

- (a) $w_0 e_0 e_2 = e_0 e_2$; (b) $w_0 s_0 w_0^{-1} = s_2$;
- (c) $w_0^3 = c$; (d) *the group generated by $\{s_0, s_2, w_0\}$ has order 24.*
- (e) $e_{i_\alpha} e_{j_\alpha} = e_{j_\alpha} e_{k_\alpha} = e_{k_\alpha} e_{i_\alpha}$.

Proof. First (a) follows from identities

$$(e_1 e_0) e_2 = s_0 s_1 s_0 s_1 s_0 e_0 e_2 = s_0 s_1 s_0 s_1 e_0 e_2 \text{ and } (e_1 e_0) e_2 = (e_1 e_2) e_0 = s_1 s_2 s_1 e_2 e_0.$$

(b) and (c) follow by direct computations. Denote the group in (d) as H . Computation shows $w_0 s_2 w_0^{-1} = c s_0 s_2$ which implies the order 8 abelian subgroup H' generated by $\{s_0, s_2, c\}$ is normal in H , and the quotient group H/H' consists of $\{[1], [w_0], [w_0^2]\}$. So (d) follows. For (e) we first prove the special case of $\alpha = 1$ then the other cases follow by conjugating action of G_{H_3} . Now the first "=" in (e) is certified by the following identity and the second one can be proved similarly. The first '=' below is by (a).

$$e_0 e_2 = w_0 e_0 e_2 w_0^{-1} = w_0 e_0 w_0^{-1} w_0 e_2 w_0^{-1} = e_2 e_{i_0}.$$

□

Remark 7.1. For $0 \leq i \leq 14$, we define

$$G^i = \{w \in G_{H_3} \mid w s_i w^{-1} = e_i\} \text{ and } H^i = \{w \in G_{H_3} \mid w e_i = e_i\}.$$

For $1 \leq \alpha \leq 5$ define $G_\alpha = \{w \in G_{H_3} \mid w(R_\alpha) = R_\alpha\}$ and $H_\alpha = \{w \in G_{H_3} \mid w e_{i_\alpha} e_{j_\alpha} = e_{i_\alpha} e_{j_\alpha}\}$.

There is $H^i \subset G^i$ and $H_\alpha \subset G_\alpha$. Since G_{H_3} acts on R transitively and $|R| = 15$, we see if s_j, s_k are the other two reflections commuting with s_i , then G^i is the order 8 group generated by $\{s_i, s_j, s_k\}$. By (2) of Theorem 7.1 below we know H^i is the order 2 group generated by s_i .

Since the action of G_{H_3} on R is transitive we have $|G_\alpha| = 24$. By (d) of above lemma we have $H_\alpha = G_\alpha$, and that G_1 is generated by $\{s_0, s_2, w_0\}$.

Lemma 7.2. $\dim B_{G_{H_3}}(\Upsilon) \leq 1045$.

Proof. Recall the set spanning $B_{G_{H_3}}(\Upsilon)$ in lemma 5.2. Consider an element $e = e_i e_j e_k$, such that i, j, k are different and every two elements in $\{i, j, k\}$ satisfy condition (4) of Definition 1.1. Notice any two of $\{e_i, e_j, e_k\}$ commutes. By above discussion $\{s_i, s_j, s_k\}$ is some class R_α . Now we have $e_{i_\alpha} e_{j_\alpha} e_{k_\alpha} = e_{j_\alpha} e_{k_\alpha} e_{i_\alpha} = \tau e_{j_\alpha} e_{k_\alpha}$, which is by (e) of Lemma 7.1. So we have proved that $B_{G_{H_3}}(\Upsilon)$ is spanned by the set

$$\Lambda = G_{H_3} \coprod \{we_i\} \coprod \{we_i e_j | s_i \perp s_j\}.$$

The relation $s_i e_i = e_i$ implies $|\{we_i\}| \leq \frac{120}{2} \times 15 = 900$. By above discussion and (e) of lemma 7.1, we know there are at most 5 kinds of $e_i e_j$ appearing in $\{we_i e_j | s_i \perp s_j\}$. For every such $e_i e_j$ there is a group $H_{i,j}$ of order 24 such that $we_i e_j = e_i e_j$ for any $w \in H_{i,j}$. So $|\{we_i e_j | s_i \perp s_j\}| \leq \frac{120}{24} \times 5 = 25$ and the lemma follows. \square

Remark 7.2. The set Λ can be presented explicitly as follows. For any i let $\{w_j^i\}_{1 \leq j \leq 60}$ be a set of representatives of left cosets of the group $\langle 1, s_i \rangle$. Let $\{w_\beta^\alpha\}_{1 \leq \beta \leq 5}$ be a set of representatives of the left cosets of H_α (see Remark 7.1) in G_{H_3} . Then

$$\Lambda = G_{H_3} \coprod \{w_j^i e_i\}_{0 \leq i \leq 14; 1 \leq j \leq 60} \coprod \{w_\beta^\alpha e_{i_\alpha} e_{j_\alpha}\}_{1 \leq \alpha \leq 5; 1 \leq \beta \leq 5}.$$

Some Irreducible Representations.

There are four 15 dimensional irreducible representations and one 5 dimensional irreducible representations of $B_{G_{H_3}}(\Upsilon)$ as follows.

The conjugating action of G_{H_3} on R is transitive. Since every element of the subgroup $G_0 = \langle s_0, s_2, c \rangle$ commutes with s_0 and $|G_0| = 8$, so G_0 is the stablizing group of this action at s_0 . G_0 has the following four one dimensional representations $\{\sigma_i\}_{0 \leq i \leq 3}$ that sending s_0 to 1.

$$\begin{aligned} (1) \sigma_0(s_0, s_2, c) &= (1, 1, 1); & (2) \sigma_1(s_0, s_2, c) &= (1, -1, 1); \\ (3) \sigma_2(s_0, s_2, c) &= (1, 1, -1); & (4) \sigma_3(s_0, s_2, c) &= (1, -1, -1). \end{aligned}$$

For every $0 \leq i \leq 3$, we have a left representation of G_{H_3} : $\text{Ind}_{G_0}^{G_{H_3}}(\sigma_i)$. They are all 15 dimensional representations whose representation spaces can be identified with a space V spanned by a basis $\{v_i\}_{0 \leq i \leq 14}$ in bijection with the set of left cosets $\{wG_0\}_{w \in G_{H_3}}$. The bijection ϕ from the second set to the first one is defined by: if $ws_0w^{-1} = s_i$ then $\phi(wG_0) = v_i$.

Now for every $0 \leq \alpha \leq 3$, we can extend every $\text{Ind}_{G_0}^{G_{H_3}}(\sigma_\alpha)$ to a representation ρ_α of $B_{G_{H_3}}(\Upsilon)$ as follows.

$$\begin{aligned} (1) \rho_\alpha(e_i)(v_i) &= \tau v_i. & (2) \rho_\alpha(e_i)(v_j) &= 0 \text{ if } i \neq j \text{ and } s_i \perp s_j. \\ (3) \rho_\alpha(e_i)(v_j) &= \sigma_\alpha(s_k)(v_j) \text{ if } i \neq j, \text{ and } s_i \text{ isn't perpendicular to } s_j \text{ such that } s_k \text{ is the} \\ & \text{unique reflection satisfying } s_k s_i s_k = s_j. \end{aligned}$$

By definition the operator $\rho_\alpha(e_i)$ is a projector to the line $\mathbb{C}v_i \subset V$ for all α . For every $0 \leq \alpha \leq 3$, define a 15×15 matrix $M^\alpha = (m_{i,j}^\alpha)$ by the identities $\rho_\alpha(e_i)(v_j) = m_{i,j}^\alpha v_i$. All entries of M^α belong to $\{\pm 1, \tau, 0\}$. By definition the diagonal elements of all M^α 's are all τ , and non-diagonal elements are all constants. So $\det M^\alpha$ is a nonzero polynomial of τ for all α .

Proposition 7.1. (1) Above definition of $\rho_\alpha(e_i)$'s extends $\text{Ind}_{G_0}^{G_{H_3}}(\sigma_\alpha)$ to a representation ρ_α of $B_{G_{H_3}}(\Upsilon)$. (2) The representation ρ_α is irreducible if and only if $\det M^\alpha \neq 0$. (3) $\rho_\alpha \not\cong \rho_\beta$ if $\alpha \neq \beta$.

Proof. Direct computation shows (1). For (2), first we observe v_i is a generator for any i since the conjugating action of G_{H_3} on R is transitive. Suppose $v \in V$ is a nonzero vector. If $\det M^\alpha \neq 0$ then there is some i such that $0 \neq \rho_\alpha(e_i)(v) \in \mathbb{C}v_i$ by definition of M^α . So v is a generator and ρ_α is irreducible. If $\det M^\alpha = 0$ then the space $\cap_{i=0}^{14} \text{Ker} \rho_\alpha(e_i)$ is nonzero. It isn't hard to see this subspace is a submodule, thus ρ_α is reducible.

For (3), suppose $\psi : V \rightarrow V$ is an $B_{G_{H_3}}(\Upsilon)$ isomorphism from ρ_α to ρ_β . By $\psi(\rho_\alpha(e_i)(v)) = \rho_\beta(e_i)(\psi(v))$, so $\psi(\text{Im} \rho_\alpha(e_i)) = \text{Im} \rho_\beta(e_i)$, which implies $\psi(v_i) = \lambda_i v_i$ for some $\lambda_i \neq 0$. Now for $w \in G_0$, on one hand we have

$$\psi(\rho_\alpha(w)(v_0)) = \psi(\sigma_\alpha(w)v_0) = \sigma_\alpha(w)\lambda_0 v_0, \text{ on the other hand}$$

$\psi(\rho_\alpha(w)(v_0)) = \rho_\beta(w)(\psi(v_0)) = \lambda_0 \sigma_\beta(w)v_0$. So we have $\sigma_\beta(w) = \sigma_\alpha(w)$ for any $w \in G_0$, which implies $\alpha = \beta$. □

There is another irreducible representation related to the action of G_{H_3} on \mathcal{R} defined as follows.

Let $U = \mathbb{C} \langle u_1, \dots, u_5 \rangle$ be a 5 dimensional vector space. Define a representation of G_{H_3} on U as: $w(u_i) = u_j$ if $w(R_i) = R_j$, for $w \in G_{H_3}$ and $1 \leq i \leq 5$.

For $0 \leq i \leq 14$, define $[i] \in \{1, \dots, 5\}$ by the relation $s_i \in R_{[i]}$. For $0 \leq i \leq 14$, $1 \leq p \leq 5$, we set

$$e_i(v_p) = \begin{cases} \tau v_p, & \text{if } s_i \in R_p; \\ v_{[i]}, & \text{if } s_i \notin R_p. \end{cases}$$

Lemma 7.3. Above action of G_{H_3} and e_i 's on U extends to a representation ρ_4 of $B_{G_{H_3}}(\Upsilon)$. This representation is irreducible if and only if $(\tau - 1)^4(\tau + 4) \neq 0$.

Proof. The first claim can be proved by direct computations. For the second one we first observe $\rho_4(e_i)$ is a projector to $\mathbb{C}u_{[i]}$ and $\rho_4(e_i) = \rho_4(e_j)$ if i, j lie in the same equivalent class. So for every $1 \leq p \leq 5$ we have a well defined projector J_p (onto $\mathbb{C}u_p$) by setting $J_p = \rho_4(e_i)$ for any $i \in R_p$. Define a 5×5 matrix $M^4 = (m_{p,q}^4)$ by setting $J_p(u_q) = m_{p,q}^4 u_p$. This matrix is clear: diagonal entries are all τ and non-diagonal entries are all 1. So $\det M^4 = (\tau - 1)^4(\tau + 4)$. An argument similar to Proposition 7.1 shows ρ_4 is irreducible if and only if $\det M^4 \neq 0$. □

Theorem 7.1. (1) If $\prod_{p=0}^4 \det M^p \neq 0$, $B_{G_{H_3}}(\Upsilon)$ is a 1045 dimensional semisimple algebra and have Λ (remark 7.2) as a basis. Notice $\prod_{p=0}^4 \det M^p$ is a polynomial of τ .
(2) For all τ , $B_{G_{H_3}}(\Upsilon)$ is a 1045 dimensional algebra having Λ as a basis.

Proof. Suppose $\rho_5, \rho_6, \dots, \rho_N$ are all irreducible representations of $B_{G_{H_3}}(\Upsilon)$ induced by the quotient map $\pi: B_{G_{H_3}}(\Upsilon) \rightarrow \mathbb{C}G_{H_3}$. They are different from ρ_0, ρ_1, \dots , and ρ_4 because e_i 's act as zero on them. We have $\sum_{i=5}^N \dim \rho_i^2 = |G_{H_3}| = 120$.

If $\prod_{p=0}^4 \det M^p \neq 0$ then ρ_1, ρ_2, \dots , and ρ_4 are all irreducible. In these cases by Wedderburn-Artin theorem we have

$$\dim B_{G_{H_3}}(\Upsilon) \geq \sum_{i=5}^N \dim \rho_i^2 + \sum_{i=0}^3 \dim \rho_i^2 + \dim \rho_4^2 = 120 + 900 + 25 = 1045.$$

Combining with lemma 7.2 we have proved (1).

For later convenience, in the set Λ we denote elements of G_{H_3} as x_1, \dots, x_{120} , denote e_1, \dots, e_{15} as x_{121}, \dots, x_{135} . Denote other elements of Λ as x_{136}, \dots, x_{1045} . Suppose first $\prod_{p=0}^4 \det M^p \neq 0$. In these case since $\{x_i\}_{1 \leq i \leq 1045}$ is a basis of $B_{G_{H_3}}(\Upsilon)$, every product $x_i x_j$ can be expanded uniquely as a linear sum of x_k 's: (*) $x_i x_j = \sum_{k=1}^{1045} f_{i,j}^k(\tau) x_k$.

We observe the following facts.

- (a) $f_{i,j}^k(\tau)$'s are all polynomial functions of τ ;
- (b) The identity (*) actually holds in $B_{G_{H_3}}(\Upsilon)$ for all τ 's.

Let $A = \mathbb{C} \langle X_1, \dots, X_{1045} \rangle$ be a vector space with a basis $\{X_1, \dots, X_{1045}\}$. Define a product on A by setting $X_i X_j = \sum_{k=1}^{1045} f_{i,j}^k(\tau) X_k$. This product make A into an associative algebra when $\prod_{p=0}^4 \det M^p \neq 0$. Combining with (a) it follows that this product is well-defined and making A into an associative algebra for all τ 's. Denote this algebra as $A(\Upsilon)$. Recall we have argued that in case of H_3 the data Υ consists of one term τ essentially. A simple check of this product shows:

- (c) $A(\Upsilon)$ is generated by $\{X_1, \dots, X_{135}\}$ for all τ 's.

(d) The map $[x_i \mapsto X_i \text{ for } 1 \leq i \leq 135]$ extends to a morphism $\phi: B_{G_{H_3}}(\Upsilon) \rightarrow A(\Upsilon)$ for all τ 's.

By (c) the morphism ϕ is surjective. Then by lemma 7.2 ϕ is an isomorphism and (2) follows. □

Cellular structures. let $(\Lambda, M, C, *)$ be the cellular structure of $\mathbb{C}B_{G_{H_3}}(\Upsilon)$. The algebra $B_{G_{H_3}}(\Upsilon)$ has a cellular structure $(\bar{\Lambda}, \bar{M}, \bar{C}, *)$ as follows.

- $\bar{\Lambda} = \Lambda \cup \{\lambda_0, \lambda_1, \dots, \lambda_4\}$. Extend the partial order in Λ by setting: $\lambda_i \prec \lambda$ for any $0 \leq i \leq 4$ and any $\lambda \in \Lambda$; $\lambda_4 \prec \lambda_i$ for $0 \leq i \leq 3$.
- Let $*$ be the involution defined in lemma 5.5.
- For $\lambda \in \Lambda$, $\bar{M}(\lambda) = M(\lambda)$; for $0 \leq i \leq 3$, $\bar{M}(\lambda_i) = \{0, 1, \dots, 14\}$; $\bar{M}(\lambda_4) = \{1, 2, \dots, 5\}$.
- For $\lambda \in \Lambda$, $S, T \in \bar{M}(\lambda)$, let $\bar{C}_{S,T}^\lambda = j(C_{S,T}^\lambda)$. Where j is the naturally injection from $\mathbb{C}G_{H_3}$ to $B_{G_{H_3}}(\Upsilon)$.

- For every $0 \leq i \leq 14$, choose $w_i \in G_{H_3}$ such that $w_i e_0 w_i^{-1} = e_i$ and $w_0 = \text{id}$. Set $J_0 = (1+s_2+s_2c+c)$, $J_1 = (1-s_2+c-cs_2)$, $J_2 = (1+s_2-cs_2-c)$, $J_3 = (1-s_2+cs_2-c)$, which are idempotents of the group algebra $\mathbb{C} \langle s_2, c \rangle$ corresponding to $\sigma_0, \dots, \sigma_3$. Then for $0 \leq \alpha \leq 3$ and $0 \leq i, j \leq 14$, set $\bar{C}_{i,j}^{\lambda_\alpha} = w_i J_\alpha e_0 w_j^{-1}$.
- As before suppose $R_\alpha = \{i_\alpha, j_\alpha, k_\alpha\}$. For $1 \leq \alpha, \beta \leq 5$ choose $w_\beta^\alpha \in W_{H_3}$ such that $w_\beta^\alpha(R_\alpha) = R_\beta$. Where the conjugating action is defined in the beginning of this section. Then set $\bar{C}_{\beta,\alpha}^{\lambda_4} = w_\beta^\alpha e_{i_\alpha} e_{j_\alpha}$.

Theorem 7.2. *Above data define a cellular structure on $B_{G_{H_3}}(\Upsilon)$.*

Proof. (C1) follows from Theorem 7.1. (C2) is proved by the following identities.

$$\begin{aligned} *(\bar{C}_{i,j}^{\lambda_\alpha}) &= *(w_j^{-1}) * (e_0) * (J^\alpha) * (w_i) = w_j e_0 J^\alpha w_i^{-1} = \bar{C}_{j,i}^{\lambda_\alpha}; \\ *(\bar{C}_{\beta,\alpha}^{\lambda_4}) &= *(e_{j_\alpha}) * (e_{i_\alpha}) * (w_\beta^\alpha) = e_{j_\alpha} e_{i_\alpha} (w_\beta^\alpha)^{-1} = (w_\beta^\alpha)^{-1} e_{i_\beta} e_{j_\beta} = w_\alpha^\beta h e_{i_\beta} e_{j_\beta} = \bar{C}_{\alpha,\beta}^{\lambda_4}. \end{aligned}$$

Where the third "=" is because $w_\beta^\alpha(R_\alpha) = R_\beta$ and $e_{i_\beta} e_{j_\beta} = e_{j_\beta} e_{k_\beta} = e_{k_\beta} e_{i_\beta}$. By Remark 7.1, there is some $h \in G_\beta$ such that the fourth "=" holds. Also by Remark 7.1 we get the fifth "=".

(C3) in cases of $\lambda_i (0 \leq i \leq 3)$ are proved by the following identities.

(1) $w \bar{C}_{i,j}^\alpha = w w_i J^\alpha e_0 w_j^{-1} = w_k (w_k^{-1} w w_i) J^\alpha e_0 w_j^{-1} = \sigma_\alpha(w_k^{-1} w w_i) \bar{C}_{k,j}^\alpha$. Where k is determined by $w_k^{-1} w w_i \in G_0$. The last "=" is because $v J^\alpha = \sigma_\alpha(v) J^\alpha$ for $v \in G_0$.

$$(2) e_l \bar{C}_{i,j}^\alpha = e_l w_i J^\alpha e_0 w_j^{-1} = e_l e_i w_i J^\alpha w_j^{-1}$$

$$= \begin{cases} 0 \text{ mod } (I^{<\lambda_\alpha}) & \text{when } s_l \perp s_i; \\ s_{l,i} e_i w_i J^\alpha w_j^{-1} = s_{l,i} w_i J^\alpha e_0 w_j^{-1} = & \\ \sigma_\alpha(w_k^{-1} s_{l,i} w_i) \bar{C}_{k,j}^\alpha \text{ mod } (I^{<\lambda_\alpha}) & \text{when } s_l \text{ isn't perpendicular to } s_i. \end{cases}$$

where $s_{l,i}$ is the unique reflection such that $s_{l,i} s_i s_{l,i} = s_l$, k is determined by $w_k^{-1} s_{l,i} w_i \in G_0$, and $I^{<\lambda_\alpha}$ is the ideal generated by $\{\bar{C}_{i,j}^\lambda\}_{\lambda < \lambda_\alpha}$.

In the case of (C4) we have

(3) $w \bar{C}_{\beta,\alpha}^{\lambda_4} = w w_\beta^\alpha e_{i_\alpha} e_{j_\alpha} = \bar{C}_{\gamma,\alpha}^{\lambda_4}$. Where γ is determined by $w w_\beta^\alpha(R_\alpha) = R_\gamma$. The last "=" is by using remark 7.1.

(4)

$$e_i \bar{C}_{\beta,\alpha}^{\lambda_4} = e_i e_{i_\beta} e_{j_\beta} w_\beta^\alpha = \begin{cases} \tau \bar{C}_{\beta,\alpha}^{\lambda_4} & \text{when } i \in \{i_\beta, j_\beta, k_\beta\}. \\ s_{i,i_\beta} e_{i_\beta} e_{j_\beta} w_\beta^\alpha = \bar{C}_{\gamma,\alpha}^{\lambda_4} & \text{otherwise.} \end{cases}$$

Where γ is determined by $i \in R_\gamma$. □

8 Canonical Presentations

8.1 Real cases

We define an algebra $B'_G(\Upsilon)$ with certain canonical presentation when G is a Coxeter group or a cyclotomic reflection group of type $G(m, 1, n)$, then prove $B'_G(\Upsilon)$ is isomorphic to $B_G(\Upsilon)$. First we do it in cases of dihedral groups.

Definition 8.1. The algebra $B'_G(\Upsilon)$ have the following presentation when G is G_n , the dihedral group of type $I_2(n)$.

TABLE 3. Presentation for $B_{G_n}(\Upsilon)$.

	$B_{G_{2k+1}}(\Upsilon)$	$B_{G_{2k}}(\Upsilon)$
<i>generators</i>	S_0, S_1, E_0, E_1	S_0, S_1, E_0, E_1
<i>relations</i>	1) $[S_0 S_1 \cdots]_{2k+1} = [S_1 S_0 \cdots]_{2k+1}$; 2) $S_0^2 = S_1^2 = 1$; 3) $S_i E_i = E_i = E_i S_i$ for $i = 0, 1$; 4) $E_i^2 = \tau_i E_i$ for $i = 0, 1$; 5) $E_0 [S_1 S_0 \cdots]_{2i-1} E_0$ $= \mu E_0$ for $1 \leq i \leq k$; 6) $E_1 [S_0 S_1 \cdots]_{2i-1} E_1$ $= \mu E_1$ for $1 \leq i \leq k$; 7) $[S_0 S_1 \cdots]_{2k} E_0$ $= E_1 [S_0 S_1 \cdots]_{2k}$; 8) $[S_1 S_0 \cdots]_{2k} E_1$ $= E_0 [S_1 S_0 \cdots]_{2k}$.	1) $[S_0 S_1 \cdots]_{2k} = [S_1 S_0 \cdots]_{2k}$; 2) $S_0^2 = S_1^2 = 1$; 3) $S_i E_i = E_i = E_i S_i$ for $i = 0, 1$; 4) $E_i^2 = \tau_i E_i$ for $i = 0, 1$; 5) $E_0 [S_1 S_0 \cdots]_{2i-1} E_0$ $= (\mu_i + \mu_{i+k}) E_0$ for $1 \leq i \leq k$; 6) $E_1 [S_1 S_0 \cdots]_{2i-1} E_1$ $= (\mu_i + \mu_{i+k}) E_1$ for $1 \leq i \leq k$; 7) $[S_1 S_0 \cdots]_{2k-1} E_0$ $= E_0 [S_1 S_0 \cdots]_{2k-1} = E_0$; 8) $[S_0 S_1 \cdots]_{2k-1} E_1$ $= E_1 [S_0 S_1 \cdots]_{2k-1} = E_1$; 9) $E_1 W E_0 = E_0 W E_1 = 0$.

Where in 9) W is any element composed by $\{S_0, S_1\}$.

Theorem 8.1. If G is a dihedral group, then $B_G(\Upsilon)$ is isomorphic to $B'_G(\Upsilon)$.

Proof. We consider the cases when G is of type $I_2(2k)$. The cases for G of type $I_2(2k+1)$ are similar and easier.

Denote the algebra $B_G(\Upsilon)$, $B'_G(\Upsilon)$ as B , B' respectively. Let j be the morphism from $\mathbb{C}G$ to B' by mapping $s_i \in G$ to $S_i \in B'$ for $i = 0, 1$. Let π be the morphism from B' to $\mathbb{C}G$ by mapping $S_i \in B'$ to $s_i \in G$, E_i to 0. There is $\pi \circ j = \text{id}_{\mathbb{C}G}$, which implies that j is injective. For saving notations we denote $j(w)$ as w for $w \in G$.

For $2 \leq 2i \leq 2k-2$, choose any $w \in G$ such that $s_{2i} = w s_0 w^{-1}$ and let $E_{2i} = w E_0 w^{-1}$. E_{2i} is well defined with no dependence on choice of w (a special case of Lemma 8.1 later). For example, choose $w = [S_1 S_0 \cdots]_{2i-1}$ so $E_{2i} = [S_1 S_0 \cdots]_{2i-1} E_0 [S_1 S_0 \cdots]_{2i-1}$. Similarly for $3 \leq 2i-1 \leq 2k-1$, define $E_{2i-1} = [S_1 S_0 \cdots]_{2i-2} E_1 [S_1 S_0 \cdots]_{2i-2} = [S_1 S_0 \cdots]_{2i-1} E_1 [S_1 S_0 \cdots]_{2i-1}$. Define a map ϕ from the set of generators of B to B' as : $\phi(T_w) = w (= j(w))$; $\phi(e_i) = E_i$ for $0 \leq i \leq 2k-1$. Then ϕ extends to a morphism from B to B' . To prove it we only need to certify that ϕ keep all the relations in Definition 1.1. The case of relation (0) is straightforward. Relation (1) is by 3) in Definition 8.1 of $B_{G_{2k}}(\Upsilon)$; (1)' is by 7), 8); (2) is by 4); 3) is by later Lemma 8.1; case of 4) doesn't arise here; (6) is by 9); 5) is by the following computations. First consider the relation for $e_{2i} e_0$. We have

$$\begin{aligned} \phi(e_{2i})\phi(e_0) &= E_{2i} E_0 = [s_1 s_0 \cdots]_{2i-1} E_0 [s_1 s_0 \cdots]_{2i-1} E_0 = s_i E_0 s_i E_0 = s_i (\mu_i + \mu_{[i+k]}) E_0 \\ &= (\mu_i s_i + \mu_{[i+k]} s_{[i+k]}) E_0 = \phi((\mu_{k(2i,0)} s_{k(2i,0)} + \mu_{k(2i,0)'} s_{k(2i,0)'}) e_0) = \phi(e_{2i} e_0). \end{aligned}$$

Relations for other $e_{2i}e_{2j}$ can be obtained by suitable conjugating action of G on above equation. The relations for $e_{2i+1}e_{2j+1}$ are similar.

There is a natural morphism $\psi : B' \rightarrow B$ by extending the correspondence $S_0 \mapsto s_0$, $S_1 \mapsto s_1$, $E_0 \mapsto e_0$, $E_1 \mapsto e_1$. The fact that ψ keep relations 1), 2) of Definition 8.1 is by (1) of Definition 1.1; 3) is by (1); 4) is by (2); 5), 6) are by (5); 7), 8) are by (1)'; 9) is by (6).

□

Suppose G_M is a finite Coxeter group with Coxeter matrix $M = (m_{i,j})_{n \times n}$. The group G_M has the following presentation:

$$\langle s_1, s_2, \dots, s_n \mid [s_i s_j \dots]_{m_{i,j}} = [s_j s_i \dots]_{m_{i,j}} \text{ for } i \neq j; s_i^2 = 1 \text{ for any } i \rangle.$$

It is well-known that G_M can be realized as a group generated by reflections in some n dimensional linear space through certain geometric representation $\rho : G \rightarrow GL(V)$. We identify G_M with its image in $GL(V)$, denote $\rho(s_i)$ as s_i . Since G_M is real, the index set of reflection hyperplanes P are in one to one correspondence with the set of reflections R . So it is convenient to denote the reflection hyperplane of $s \in R$ as H_s and write e_i in the Definition 1.1 as e_s . In the following we denote G_M as G . For $w \in G$, any expression $w = s_{i_1} s_{i_2} \dots s_{i_r}$ with minimal length is called a reduced form of w , and define the length of w as $l(w) = r$. Above definition of $B'_G(\Upsilon)$ when G is a dihedral group invoke the following definition of $B'_{G_M}(\Upsilon)$.

Definition 8.2. For any Coxeter matrix $M = (m_{i,j})_{n \times n}$, the algebra $B'_{G_M}(\Upsilon)$ is defined as follows. Denote τ_{s_i} in Υ as τ_i . If we don't give range for an index then it means "for all". The generators are $S_1, \dots, S_n, E_1, \dots, E_n$. The relations are

- | | |
|--|---|
| 1) $S_i^2 = 1$; | 8) $E_i w E_j = 0$ for any word w |
| 2) $[S_i S_j \dots]_{m_{i,j}} = [S_j S_i \dots]_{m_{i,j}}$; | composed from $\{S_i, S_j\}$ If $m_{i,j} = 2k > 2$; |
| 3) $S_i E_i = E_i = E_i S_i$; | 9) $E_i [S_j S_i \dots]_{2l-1} E_i = (\mu_s + \mu_{s'}) E_i$ |
| 4) $E_i^2 = \tau_i E_i$; | for $1 \leq l \leq k$, If $m_{i,j} = 2k > 2$. |
| 5) $S_i E_j = E_j S_i$ if $m_{i,j} = 2$; | 10) $E_i [S_j S_i \dots]_{2l-1} E_i = \mu_{s_\epsilon} E_i$ |
| 6) $E_i E_j = E_j E_i$ if $m_{i,j} = 2$; | for $1 \leq l \leq k$, if $m_{i,j} = 2k + 1$; |
| 7) $[S_j S_i \dots]_{2k-1} E_i = E_i [S_j S_i \dots]_{2k-1} = E_i$, | Where $\epsilon = i(j)$ if l is odd (even). |
| if $m_{i,j} = 2k > 2$; | 11) $[S_i S_j \dots]_{2k} E_i = E_j [S_i S_j \dots]_{2k}$ If $m_{i,j} = 2k + 1$. |

Remark 8.1. If M is irreducible and of simply laced type, i.e., $m_{i,j} \in \{1, 2, 3\}$, we can set all $\mu_s = 1$ by Lemma 5.3, so above definition coincide with the definition of simply laced Brauer algebras in [CFW]. Above definition includes the case $m_{i,j} = \infty$: in that case there are no other relations between S_i, S_j, E_i, E_j except (1), (3), (4).

Let M be as above. The Artin group A_M has the following presentation.

$$\langle \sigma_1, \sigma_2, \dots, \sigma_n \mid [\sigma_i \sigma_j \dots]_{m_{i,j}} = [\sigma_j \sigma_i \dots]_{m_{i,j}} \text{ for } i \neq j \rangle.$$

Here we denote A_M as A . Let A^+ be the monoid generated with the same set of generators and relations. Let $J : A^+ \rightarrow A$ be the natural morphism of monoids. It is proved that J is injective for all Artin groups Garside [Ga] Brieskorn-Saito [BS] Paris [Pa] . The following theorem is well known.

Theorem 8.2. *For any $w \in G$, suppose $l(w) = r$ and let $s_{i_1} \cdots s_{i_r}$ and $s_{j_1} \cdots s_{j_r}$ be two reduced forms of w , then in A^+ we have $\sigma_{i_1} \cdots \sigma_{i_r} = \sigma_{j_1} \cdots \sigma_{j_r}$.*

So there is a well defined injective map $\tau : G \rightarrow A^+$ as follows. For $w \in G$, let $s_{i_1} \cdots s_{i_k}$ be a reduced form of w and let $\tau(w) = \sigma_{i_1} \cdots \sigma_{i_k}$. Denote the natural map from A^+ to G extending $\sigma_i \mapsto s_i$ as π . In A^+ we denote $b \prec c$ if there is $a \in A^+$ such that $ab = c$. This define a partial order for A^+ . Here is an important result in Artin group theory.

Theorem 8.3 ([BS], [Ga]). *For $a \in A^+$, if $\sigma_i \prec a$, $\sigma_j \prec a$, then $[\cdots \sigma_j \sigma_i]_{m_{i,j}} \prec a$.*

Now we can prove the following lemma.

Lemma 8.1. *Suppose G acts on a set S . Suppose a subset $\{v_1, \dots, v_n\} \subset S$ satisfy*

- (1) If $m_{i,j} = 2k + 1$, then $[s_i s_j \cdots]_{2k}(v_i) = v_j$; (3) If $m_{i,j} = 2$, then $s_i(v_j) = v_j$;
- (2) If $m_{i,j} = 2k$, then $[s_i s_j \cdots]_{2k-1}(v_j) = v_j$; (4) $s_i(v_i) = v_i$.

Then an identity $ws_i w^{-1} = s_j$ in G implies $w(v_i) = v_j$.

Proof. We prove it by induction on $l(w)$. When $l(w) = 0$ it is evident. Suppose the lemma is true when $l(w) < k$ and suppose we have an identity $ws_i w^{-1} = s_j$ where $l(w) = k$. If $l(ws_i) = l(w) - 1$, let $w' = ws_i$. Since $w' s_i (w')^{-1} = ws_i w^{-1} = s_j$, by induction we have $w'(v_i) = v_j$. Which implies $w(v_i) = w'(v_i) = v_j$ by (4). Now suppose $l(ws_i) = l(w) + 1$. Let $s_{i_1} \cdots s_{i_k}$ be a reduced form of w . We have $s_{i_1} \cdots s_{i_k} s_i = s_j s_{i_1} \cdots s_{i_k}$. Because both sides of the identity are reduced forms, by Theorem 8.2 we have $\sigma_{i_1} \cdots \sigma_{i_k} \sigma_i = \sigma_j \sigma_{i_1} \cdots \sigma_{i_k} = \tau(ws_i)$. From the condition $l(ws_i) = l(w) + 1$ we know $i_k \neq i$, so by Theorem 8.3 we have $[\cdots \sigma_{i_k} \sigma_i]_{m_{i_k,i}} \prec \tau(ws_i)$. So $\tau(ws_i) = a[\cdots \sigma_{i_k} \sigma_i]_{m_{i_k,i}}$ for some $a \in A^+$. Denote $\pi(a)$ as w' , and $\pi([\cdots \sigma_i \sigma_{i_k}]_{m_{i_k,i}-1})$ as u . So $w = w'u$. An argument of length shows $l(w') = l(w) - l(u)$. There is $us_i u^{-1} = s_{i_k}$, and by (1), (2), (3) we have $u(v_i) = v_{i_k}$. So $w' s_{i_k} (w')^{-1} = ws_i w^{-1} = s_j$. By induction we have $w'(v_{i_k}) = v_j$ which implies $w(v_i) = w'u(v_i) = v_j$. □

Theorem 8.4. *When G_M is finite then $B_{G_M}(\Upsilon) \cong B'_{G_M}(\Upsilon)$.*

The proof is still by constructing a morphism from $B'_G(\Upsilon)$ to $B_G(\Upsilon)$ and a morphism back, then show they are the inverse of each other. The following lemma is well-known Humphreys [Hu].

Lemma 8.2. *Suppose G is a finite Coxeter group. Then*

(1) For any i, j , if $w \in G$ fix $H_{s_i} \cap H_{s_j}$ point-wise, then w lies in the subgroup generated by s_i, s_j .

(2) For any two different $s, s' \in R$, there are $w \in G, 1 \leq i < j \leq n$ such that $w(H_s \cap H_{s'}) = H_{s_i} \cap H_{s_j}$. So $ws w^{-1}$ and $ws' w^{-1}$ lie in the subgroup generated by s_i and s_j .

Lemma 8.3. Define a map ϕ by setting $\phi(S_i) = s_i \in G, \phi(E_i) = e_{s_i}$ for $1 \leq i \leq n$, then ϕ extends to a morphism from $B'_G(\Upsilon)$ to $B_G(\Upsilon)$.

Proof. We need to certify that ϕ preserves all relations of in Definition 8.2. The facts ϕ preserves 1) \sim 8) and 11) are easy to see. Notice in 9) and 10) only two indices i, j are involved. So we can use lemma 5.4 and lemma 8.2 to reduce these cases to dihedral cases, which are proved in Theorem 8.1. \square

It isn't hard to see that the morphism J from $\mathbb{C}G$ to $B'_G(\Upsilon)$ by sending s_i to S_i is injective. So for $w \in G$ we can identify $J(w)$ with w for convenience. Denote the imbedding image of R in $B'_G(\Upsilon)$ as R' . Let $E' = \{wE_i w^{-1}\}_{w \in G, 1 \leq i \leq n}$, $E = \{e_s\}_{s \in R}$. By definition of $B'_G(\Upsilon)$, the conjugating action of G on E' satisfies conditions in Lemma 8.1. So the map $e_i \mapsto E_i$ ($1 \leq i \leq n$) extends uniquely to a G -equivariant surjective map $\varphi : E \rightarrow E'$. Define a map $\psi : E \cup R \rightarrow E' \cup R'$ by $\psi(e_s) = \varphi(e_s)$, $\psi(s) = s$. We have the following lemma.

Lemma 8.4. The map ψ extends to a morphism from $B_G(\Upsilon)$ to $B'_G(\Upsilon)$. Still denote it as ψ .

Proof. We need to show that ψ satisfies all relations in Definition 5.1. The cases for relation (0), (1) and (2) is evident. The case for relation (1)' follows from relation (4) of Definition 8.3. Case of relation (3) is by definition of φ . For relations (3) to (6), we can reduce these cases by (2) of lemma 8.2 and lemma 5.4 to cases of dihedral groups, which are proved in Theorem 8.1. \square

By definition ψ and ϕ are apparently the inverse of each other, so Theorem 8.4 is proved.

8.2 The Cyclotomic $G(m, 1, n)$ cases

Let G be the cyclotomic pseudo reflection group of type $G(m, 1, n)$. As in BMR[BMR], let V be a n -dimensional complex linear space with a positive definite Hermitian metric \langle, \rangle , let $\{v_1, \dots, v_n\}$ be a orthonormal base. Then G can be imbedded in $U(V)$. It's image consists of monomial matrices whose nonzero entries are m -th roots of unit. Here we give a concise description of some facts of G without proof. Suppose (z_1, \dots, z_n) is the coordinate system corresponding to $\{v_1, \dots, v_n\}$. Let $\xi = \exp(\frac{2\pi\sqrt{-1}}{m})$. For $i \neq j$, $0 \leq a \leq m-1$, define $H_{i,j;a} = \ker(z_i - \xi^a z_j) = (v_i - \xi^a v_j)^\perp$. Define $H_i = \ker(z_i) = (v_i)^\perp$. Then $H_{i,j;a} = H_{j,i;-a}$.

Let $s_{i,j;a} \in U(V)$ be the unique reflection fixing every points in $H_{i,j;a}$. Let s_i be the pseudo reflection defined by: $s_i(v_i) = \xi v_i$; $s_i(v_j) = v_j$ for $j \neq i$. Then the set \mathcal{A} of reflection

hyperplanes of G is $\{H_{i,j;a}\}_{i < j; 0 \leq a \leq m-1} \cup \{H_i\}_{1 \leq i \leq n}$. The set of pseudo reflections R of G is

$$\{s_{i,j;a}\}_{i < j; 0 \leq a \leq m-1} \amalg (\amalg_{k=1}^{m-1} \{s_i^k\}_{1 \leq i \leq n}).$$

Above notation gives a decomposition of R into conjugacy classes. Now we have a look at the algebra $B_G(\Upsilon)$. The data Υ essentially consists of $\mu, \mu_1, \dots, \mu_{m-1}, \tau_0, \tau_1$. Where $\mu_{s_{i,j;a}} = \mu$, $\mu_{s_i^k} = \mu_k$, $\tau_{H_{i,j;a}} = \tau_1$, $\tau_{H_i} = \tau_0$. As in the real case, we define the following algebra $B'_G(\Upsilon)$ with canonical presentation.

Definition 8.3. *The algebra $B'_G(\Upsilon)$ is generated by $S_0, S_1, \dots, S_{n-1}, E_0, E_1, \dots, E_{n-1}$ with the following relations. Where in 15) W is any word composed from S_0, S_1 .*

- 1) $S_0^m = S_i^2 = \text{id} (1 \leq i \leq n-1)$;
- 2) $S_0 S_1 S_0 S_1 = S_1 S_0 S_1 S_0$;
- 3) $S_i S_j = S_j S_i (|i-j| \geq 2)$;
- 4) $S_i S_{i+1} S_i = S_{i+1} S_i S_{i+1} (1 \leq i \leq n-2)$;
- 5) $E_0^2 = \tau_0 E_0$;
- 6) $E_i^2 = \tau_1 E_i (1 \leq i \leq n-1)$;
- 7) $S_1 S_0 S_1 E_0 = E_0 S_1 S_0 S_1 = E_0$;
- 8) $S_i S_{i+1} E_i = E_{i+1} S_i S_{i+1} (1 \leq i \leq n-2)$;
- 9) $S_i E_j = E_j S_i (|i-j| \geq 2)$;
- 10) $(S_0)^i S_1 (S_0)^i (E_1) = E_1 (S_0)^i S_1 (S_0)^i (0 \leq i \leq m-1)$;
- 11) $S_i E_i = E_i = E_i S_i (0 \leq i \leq n-1)$;
- 12) $E_1 S_0^i E_1 = \mu_i E_1 (0 \leq i \leq m-1)$;
- 13) $E_0 S_1 E_0 = (m-1) \mu E_0$;
- 14) $E_i E_j = E_j E_i (|i-j| \geq 2)$;
- 15) $E_0 W E_1 = E_1 W E_0 = 0$;
- 16) $E_i E_{i+1} = \mu S_i S_{i+1} S_i E_{i+1} (1 \leq i \leq n-2)$.

Theorem 8.5. *The algebra $B'_G(\Upsilon)$ is isomorphic to $B_G(\Upsilon)$.*

Proof. The strategy of proof of this theorem is the same as for the real case. We construct a morphism Φ from $B'_G(\Upsilon)$ to $B_G(\Upsilon)$ and a morphism Ψ in reverse direction. The morphism Φ is constructed by setting $\Phi(S_0) = s_1$; $\Phi(S_i) = s_{i,i+1;0}$ for $i \geq 1$; $\Phi(E_0) = e_1$; $\Phi(E_i) = e_{i,i+1,0}$ for $i \geq 1$. It isn't hard to certify that Φ satisfying all relations in Definition 8.3, so Φ can extend to a morphism. To define Ψ , the main step is still the definition of $\Psi(e_i)$ and $\Psi(e_{i,j;a})$. The following lemma shows there are well defined elements F_i 's and $F_{i,j;a}$'s such that if we set $\Psi(w) = w$, $\Psi(e_i) = F_i$, $\Psi(e_{i,j;a}) = F_{i,j;a}$, Then Ψ can extend to a morphism from $B_G(\Upsilon)$ to $B'_G(\Upsilon)$ by certifying that it all relations in Definition 1.1. The proof is almost the same as in proof of Theorem 8.5 so we skip it. □

Construction of F_i and $F_{i,j;a}$ The following lemma is similar to Lemma 8.1.

Lemma 8.5. *Let G be the pseudo reflection group of type $G(m, 1, n)$. Suppose G acts on a set S and suppose there is a subset $\{v_0, v_1, \dots, v_{n-1}\} \subset S$ such that:*

- (1) $(S_0)^i S_1 (S_0)^i (v_1) = v_1$ for any i ;
- (2) $S_1 (S_0)^i S_1 (v_0) = v_0$ for any i ,
- (3) $S_i (v_j) = v_j$ if $|i-j| \geq 2$,
- (4) $S_i S_{i+1} (v_i) = v_{i+1}$ for $i \geq 1$,
- (5) $S_i S_{i-1} (v_i) = v_{i-1}$ for $i \geq 2$,
- (6) $S_i (v_i) = v_i$.

Then the identity $w(\mathbb{H}_i) = \mathbb{H}_j$ implies $w(v_i) = v_j$, where $w \in G$, and S_i 's are generators of G as in Proposition 8.1. For convenience here we denote H_1 as \mathbb{H}_0 , $H_{i,i+1;0}$ as \mathbb{H}_i for $i \geq 2$.

Proof. In this case instead of using Artin monoid we prove it by direct computation. Let $\bar{v}_i = S_{i-1} \cdots S_1(v_0)$ for $1 \leq i \leq n$; $\bar{v}_{i,j}^a = (S_{j-1} \cdots S_1 S_0 S_1 \cdots S_{j-1})^a S_{j-1} \cdots S_{i+1}(v_i)$ for $i < j-1$; $\bar{v}_{i,i+1}^a = (S_i \cdots S_0 \cdots S_i)^a(v_i)$. The following identities show that the set $\{\bar{v}_i\}_{1 \leq i \leq n} \cup \{\bar{v}_{i,j}^a\}_{i < j}$ is closed under the action of G , and the map $J : \mathcal{A} \rightarrow \{\bar{v}_i\}_{1 \leq i \leq n} \cup \{\bar{v}_{i,j}^a\}_{i < j} : H_{i,j;a} \mapsto \bar{v}_{i,j}^a; H_i \mapsto \bar{v}_i$ is G equivariant. Thus the lemma is proved.

- (a) $S_0(\bar{v}_i) = S_{i-1} \cdots S_2 S_0 S_1(v_0) = S_{i-1} \cdots S_2 S_1 \cdot S_1 S_0 S_1(v_0) = S_{i-1} \cdots S_2 S_1(v_0) = \bar{v}_i$.
- (b) $S_0(\bar{v}_{1,i}^a) = S_0(S_{i-1} \cdots S_0 \cdots S_{i-1})^a S_{i-1} \cdots S_2(v_1) = (S_{i-1} \cdots S_0 \cdots S_{i-1})^a S_{i-1} \cdots S_2 S_0(v_1) = (S_{i-1} \cdots S_0 \cdots S_{i-1})^{a-1} S_{i-1} \cdots S_1 S_0 S_1 S_0(v_1) = (S_{i-1} \cdots S_0 \cdots S_{i-1})^{a-1} S_{i-1} \cdots S_2(v_1) = \bar{v}_{1,i}^{a-1}$.
- (c) If $i \geq 2$, $S_0(\bar{v}_{i,j}^a) = S_0(S_{j-1} \cdots S_1 S_0 S_1 \cdots S_{j-1})^a S_{j-1} \cdots S_{i+1}(v_i) = (S_{j-1} \cdots S_1 S_0 S_1 \cdots S_{j-1})^a S_{j-1} \cdots S_{i+1} S_0(v_i) = \bar{v}_{i,j}^a$.
- (d) For $i \geq 1$, $S_i(\bar{v}_i) = S_i S_{i-1} \cdots S_1(v_0) = \bar{v}_{i+1}$.
- (e) $S_i(\bar{v}_{i+1}) = \bar{v}_i$ for $i \geq 1$. (Equivalent to (d))
- (f) $S_i(\bar{v}_j) = \bar{v}_j$ if $i \neq 0$ and $j \neq i, i+1$.
 $j \neq i, i+1 \Leftrightarrow i > j$ or $i < j-1$. If $i > j$ then $S_i(\bar{v}_j) = S_i S_{j-1} \cdots S_1(v_0) = S_{j-1} \cdots S_1 S_i(v_0) = \bar{v}_j$; If $i < j-1$ then $S_i(\bar{v}_j) = S_i S_{j-1} \cdots S_1(v_0) = S_{j-1} \cdots S_i S_{i+1} S_i \cdots S_1(v_0) = S_{j-1} \cdots S_{i+1} S_i S_{i+1} \cdots S_1(v_0) = S_{j-1} \cdots S_1 S_{i+1}(v_0) = \bar{v}_j$.
- (g) $S_i(\bar{v}_{i,l}^a) = \bar{v}_{i+1,l}^a$ if $l \geq i+2$.

First we have

$$\begin{aligned} S_i(S_{l-1} \cdots S_0 \cdots S_{l-1}) &= S_{l-1} \cdots S_i S_{i+1} S_i \cdots S_0 \cdots S_{l-1} \\ &= S_{l-1} \cdots S_{i+1} S_i S_{i+1} \cdots S_0 \cdots S_{l-1} = S_{l-1} \cdots S_0 \cdots S_{i+1} S_i S_{i+1} \cdots S_{l-1} \\ &= (S_{l-1} \cdots S_0 \cdots S_{l-1}) S_i. \end{aligned}$$

So

$$\begin{aligned} S_i(\bar{v}_{i,l}^a) &= S_i(S_{l-1} \cdots S_0 \cdots S_{l-1})^a S_{l-1} \cdots S_{i+1}(v_i) = (S_{l-1} \cdots S_0 \cdots S_{l-1})^a S_i S_{l-1} \cdots S_{i+1}(v_i) \\ &= (S_{l-1} \cdots S_0 \cdots S_{l-1})^a S_{l-1} \cdots S_{i+2} S_i S_{i+1}(v_i) \\ &= (S_{l-1} \cdots S_0 \cdots S_{l-1})^a S_{l-1} \cdots S_{i+2}(v_{i+1}) = \bar{v}_{i+1,l}^a. \end{aligned}$$

$$(h) S_i(\bar{v}_{l,i}^a) = \bar{v}_{l,i+1}^a \text{ for } l < i.$$

$$\begin{aligned} S_i(\bar{v}_{l,i}^a) &= S_i(S_{i-1} \cdots S_0 \cdots S_{i-1})^a S_{i-1} \cdots S_{l+1}(v_l) \\ &= (S_i \cdots S_0 \cdots S_i)^a S_i S_{i-1} \cdots S_{l+1}(v_l) = \bar{v}_{l,i+1}^a. \end{aligned}$$

$$(i) S_i(\bar{v}_{k,l}^a) = \bar{v}_{k,l}^a \text{ if } \{k, l\} \cap \{i, i+1\} = \emptyset.$$

$$S_i(\bar{v}_{k,l}^a) = S_i(S_{l-1} \cdots S_0 \cdots S_{l-1})^a S_{l-1} \cdots S_{k+1}(v_k)$$

□

Let $\mathbb{E}' = \{w E_i w^{-1}\}_{w \in G, 0 \leq i \leq n-1}$. G acts on \mathbb{E}' by conjugation. This action satisfies the conditions of Lemma 8.5 if let $\{E_0, E_1, \dots, E_{n-1}\}$ to be $\{v_0, v_1, \dots, v_{n-1}\}$ in the lemma. By this lemma there is a unique G -equivariant map $F : \mathcal{A} \rightarrow \mathbb{E}'$ such that $F(H_1) = E_0$, $F(H_{i,i+1;0}) = E_i$ for $1 \leq i \leq n-1$. Define $F_i = F(H_i)$ for $1 \leq i \leq n$, and $F_{i,j;a} = F(H_{i,j;a})$.

By comparing Definition 2.1 (of the cyclotomic Brauer algebra) with Definition 8.3, we have the following theorem.

Theorem 8.6. *In the data Υ if $\mu = 1$, $\mu_a = \sigma_a$ ($1 \leq a \leq m-1$) and $\sigma_0 = \tau_1$, then*

(1) *Set a map Φ by : $S_i \mapsto s_i$ ($1 \leq i \leq n-1$), $S_0 \mapsto t_1$, $E_i \mapsto e_i$ ($1 \leq i \leq n-1$), $E_0 \mapsto 0$, then Φ extends to a surjective morphism from $B'_G(\Upsilon)$ to $\mathcal{B}_{m,n}(\delta)$ and $\ker \Phi = \langle E_0 \rangle$, the idea generated by E_0 .*

(2) *Set a map Ψ by : $s_i \mapsto S_i$ ($1 \leq i \leq n-1$), $t_i \mapsto S_{i-1} \cdots S_1 t_1 S_1 \cdots S_{i-1}$ ($1 \leq i \leq n$), $e_i \mapsto E_i$ ($1 \leq i \leq n-1$), then Ψ extends to an morphism from $\mathcal{B}_{m,n}(\delta)$ to $B'_G(\Upsilon)$. We have $\Phi \circ \Psi = \text{id}$, so $B'_G(\Upsilon) \cong \mathcal{B}_{m,n}(\delta) \oplus \langle E_0 \rangle$.*

9 Conclusions

If we take off the relation $(1)'$ in Definition 1.1, we obtain an algebra bigger than $B_G(\Upsilon)$. Denote this algebra as $\bar{B}_G(\Upsilon)$. It is easy to see $\bar{B}_G(\Upsilon)$ coincide with $B_G(\Upsilon)$ if G is a simply laced Coxeter group. The algebra $\bar{B}_G(\Upsilon)$ also satisfy hypothesis 1 and 2 in Section 1. We can prove $\bar{B}_G(\Upsilon)$ is finite dimensional if G is a finite group. In general, $\bar{B}_G(\Upsilon)$ has $B_G(\Upsilon)$ as a genuine quotient, thus contain more irreducible representations.

We can ask the following questions. If $B_G(\Upsilon)$ are cellular, or generically semisimple, or have invariant dimension for any finite G ? Does $B_G(\Upsilon)$ has affine cellular structure in the sense of Konig and Xi [KX] when G is an affine Coxeter group? How to deform $B_G(\Upsilon)$ by using the associated KZ connection?

In [CGW1] the authors mentioned the perspective of application of generalized BMW algebras in representation theory. Beside of it we'd like to mention that through analysis those Brauer type algebras we can obtain some new flat G -invariant connections on the complementary spaces M_G , thus obtain some new representations of the corresponding Artin group or complex braid group, just as the case of H_3 type. In the same time by solving the equations of flat sections we would encounter with some new fuchs equations on M_G and some new hypergeometric type functions.

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